

Isotropic-nematic behaviour of hard rigid rods: a percolation theoretic approach

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Abstract: Needles at different orientations are placed in an i.i.d. manner at points of a Poisson point process on \mathbb{R}^2 of density λ . Needles at the same direction have the same length, while needles at different directions maybe of different lengths. We study the geometry of a finite cluster when needles have only two possible orientations and when needles have only three possible orientations. In both these cases the asymptotic shape of the finite cluster as $\lambda \rightarrow \infty$ is shown to consists of needles only in two directions. In the two orientations case the shape does not depend on the orientation but just on the i.i.d. structure of the orientations, while in the three orientations case the shape depend on all the parameters, i.e. the i.i.d. structure of the orientations, the lengths and the orientations of the needles.

1 Introduction

Zwanzig (1963) studied a system of non-overlapping hard rods in the continuum, where the orientations (states) of the rods were restricted to a finite set. Here he observed that as the density of rods increased a phase transition occurred from an isotropic phase, where the rods are placed ‘chaotically’, to a nematic phase, where the rods are oriented in a fixed direction. This study was a continuation of a study initiated by Onsager [1949] where he showed that a system of thin cylindrical molecules in a solution undergoes a similar phase transition in high density. Flory [1956] studied the hard rod problem on a lattice, allowing the rods to have arbitrary orientations. Using mean-field techniques, he obtained such an isotropic–nematic phase transition.

Lately there has been a considerable interest among physicists in this model, with hard rods being renamed as hard needles. This interest is kindled by the connection between the entropic properties and the the phases of the system (see, e.g., Varga, Gurin and Quintana-H [2009], Gurin and Varga [2011] and Dhar, Rajesh and Stilck [2011] and references therein).

Our study, for the 2-state and the 3-state Zwanzig model is percolation theoretic. While we consider *overlapping* hard needles our results show that in the high density case, the geometry

of the needles is such that there is exactly one needle in one orientation which binds tightly the remaining needles in the other direction, thereby giving the nematic phase.

In the language of stochastic geometry, the needles form a Boolean fibre process (see e.g., Hall [1990], Stoyan Kendall and Mecke [1995]). In the case when the centres of the needles are placed according to a homogenous Poisson point process of density λ , the overlapping needles form a percolating cluster and the system displays phase transition (see Roy [1991]) as the density increases from a regime which does not admit an unbounded connected component of needles to one where such a component exists.

In this paper we study the structure of finite connected components in a high density supercritical regime. We establish the nematic behaviour as observed by Zwanzig by showing that any finite cluster consists of all but one needle bunched together in a given direction, and the other needle providing the connectivity by lying across these oriented needles. Needles of which direction and which length are preferred in such a finite cluster depend on the parameters of the process.

We first study the 2-state Zwanzig model where the needles are placed according to a Poisson point process of density λ , with needles being of two distinct orientations and needles of the same orientation being of the same length but needles of different orientations allowed to be of different lengths. In this case we reaffirm the phase transition observed in the non-overlapping hard needles model by showing that in this percolating model, a finite cluster comprising of m needles, for high density λ and for m large, typically consists of $m - 1$ needles of one orientation with only one needle in the other orientation connecting them to form a cluster. The choice of the orientation depends on which orientation is more probable, and not on either the angle of orientation or the length of the sticks. In addition, an interesting observation is that if $p_{\lambda,m}(k, l)$ denotes the probability that in a cluster of size m there are k needles of one orientation and l needles of the other orientation, then in the situation when $(k/m) \rightarrow s \in [0, 1]$ as $m \rightarrow \infty$ and when each of the directions is equally likely, we have

$$\lim_{\substack{m \rightarrow \infty \\ (k/m) \rightarrow s}} \frac{1}{m} \lim_{\lambda \rightarrow \infty} \log p_{\lambda,m}(k, \ell) = s \log s + (1 - s) \log(1 - s).$$

Thus, for $s = 0$ and $s = 1$, we have the minimal entropic phenomenon where one stick in a particular orientation binds together tightly the remaining sticks in the other orientation; and as s tends to $1/2$ we have the maximal entropic phenomenon of equal number of sticks being present in either direction, however here also they are tightly bound.

In the 3-state Zwanzig model, where three distinct orientations of the needles are allowed, the affine invariance of the model breaks down, and for high density λ , the finite clusters consist of sticks in only two directions, with the surviving directions being dependent on both the angles and the lengths of the needles in different orientations as well as the probabilities of choosing needles in different orientations. We also study, in some situations, the equivalent of $p_{\lambda,m}(k, l)$ in this case. Although the result is not as explicit as the entropy-like expression

in the case of needles with the 2-state Zwanzig model, nonetheless it provides some insight in the nematic phase of the 3-state Zwanzig model.

The paper is organised as follows:– in the next section we present the details of the model as well as the statements of our results and in Sections 3 and 4 we prove the results.

2 The model and statement of results

2.1 Notation

Let $\mathcal{R} = \mathbb{R}^2 \times [0, \pi) \times (0, \infty)$, and

$$\mathcal{M} = \mathcal{M}(\mathcal{R}) := \{\xi = \{\xi_i, i \in \mathbb{N}\} : \xi_i = (x_i, \theta_i, r_i) \in \mathcal{R}\}.$$

For $(x, \theta, r) \in \mathcal{R}$, $S(x, \theta, r) = \{x + ue_\theta, u \in [-r, r]\}$ is the needle with centre x , angle θ and length $2r$, where $e_\theta = (\cos \theta, \sin \theta)$. We define the collection of needles for $\xi \in \mathcal{M}$ as $\mathcal{S}(\xi) = \{S(x, \theta, r) : (x, \theta, r) \in \xi\}$.

We say two needles S and S' are connected and write $S \overset{\xi}{\leftrightarrow} S'$ if there exist needles $S_1, S_2, \dots, S_k \in \mathcal{S}(\xi)$ such that $S \cap S_1 \neq \emptyset$, $S' \cap S_k \neq \emptyset$ and $S_i \cap S_{i+1} \neq \emptyset$ for every $i = 1, 2, \dots, k-1$. If $\mathcal{S}(\xi)$ contains a needle S_0 centred at the origin $\mathbf{0}$, we denote by $C_0(\xi)$ the cluster of needles containing S_0 , i.e.

$$C_0(\xi) = \{y \in S : S \in \mathcal{S}(\xi), S \overset{\xi}{\leftrightarrow} S_0\}.$$

We put $C_0(\xi) = \emptyset$ if $\mathcal{S}(\xi)$ does not contain any needle with centre $\mathbf{0}$; however for our results we take a typical point of the Poisson process to be the origin so as to exclude the possibility of $C_0 = \emptyset$.

Let ρ be the Radon measure on \mathcal{R} defined by

$$\rho(dx d\theta dr) = dx \sum_{j=1}^d p_j \delta_{\alpha_j}(d\theta) \delta_{R_j}(dr), \quad (2.1)$$

where $\alpha_1 = 0 < \alpha_2 < \alpha_3 < \dots < \alpha_d < \pi$, $p_j \geq 0$, $\sum_{j=1}^d p_j = 1$, $R_j > 0$, $j = 1, 2, \dots, d$ and δ_* denotes the usual Dirac delta measure. We denote by μ_ρ the Poisson point process on $\mathcal{M}(\mathcal{R})$ with intensity measure ρ . Let

$$\Gamma_0 := \{\xi \in \mathcal{M} : (\mathbf{0}, \alpha_j, R_j) \in \xi \text{ for some } j = 1, 2, \dots, d\}. \quad (2.2)$$

For $w_i = (x_i, \theta_i, r_i)$, $i = 1, 2, \dots, m$, let

$$\mathbf{w}_m := (w_1, w_2, \dots, w_m), \{\mathbf{w}_m\} := \{w_1, w_2, \dots, w_m\}, C_0(\mathbf{w}_m) := C_0(\{\mathbf{w}_m\}). \quad (2.3)$$

For $\mathbf{k} = (k_1, k_2, \dots, k_d) \in (\mathbb{N} \cup \{0\})^d$, we denote by $\Lambda(\mathbf{k})$ the set of clusters containing exactly $|\mathbf{k}| = \sum_{j=1}^d k_j$ needles with k_j needles at an orientation α_j , $j = 1, 2, \dots, d$.

Let $\mathbf{0}$ be the origin and x a point in \mathbb{R}^2 . Let e_θ be the vector $(\cos \theta, \sin \theta)$. The vector $x \in \mathbb{R}^2$ can be represented in the bases e_θ, e_ϕ spanning \mathbb{R}^2 as $x = x^\theta(\theta, \phi)e_\theta + x^\phi(\theta, \phi)$, where $x^\theta(\theta, \phi)$ is the length of the projection of x on the e_θ axis and $x^\phi(\theta, \phi)$ is the length of the projection of x on the e_ϕ axis. Writing

$$h_\alpha(x) = \frac{x^\alpha(\alpha, \beta)}{\sin \beta}, \quad h_\beta(x) = \frac{x^\beta(\alpha, \beta)}{\sin \alpha} \text{ and } h_0(x) = h_\alpha(x) + h_\beta(x),$$

we see that

$$\begin{aligned} x^\alpha(\alpha, \beta) &= h_\alpha(x) \sin \beta, & x^\beta(\alpha, \beta) &= h_\beta(x) \sin \alpha, \\ x^0(0, \alpha) &= h_\beta(x) \sin(\beta - \alpha), & x^0(0, \alpha) &= h_0(x) \sin \beta, \\ x^0(0, \beta) &= h_\alpha(x) \sin(\beta - \alpha), & x^0(0, \beta) &= h_0(x) \sin \alpha. \end{aligned}$$

For $R_\alpha, R_\beta > 0$ and $\mathbf{x}_m = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^2)^m$, we define the following regions:-

$$\begin{aligned} B_{R_\alpha, R_\beta}^{\alpha, \beta} &:= \{x^\alpha(\alpha, \beta)e_\alpha + x^\beta(\alpha, \beta)e_\beta : (x^\alpha, x^\beta) \in [-R_\alpha, R_\alpha] \times [-R_\beta, R_\beta]\}, \\ B_{R_\alpha, R_\beta}^{\alpha, \beta}(x) &:= B_{R_\alpha, R_\beta}^{\alpha, \beta} + x, \quad x \in \mathbb{R}^2, \\ B_{R_\alpha, R_\beta}^{\alpha, \beta}(\mathbf{x}_m) &:= \bigcup_{j=1}^m B_{R_\alpha, R_\beta}^{\alpha, \beta}(x_j). \end{aligned}$$

2.2 Needles of two types

In this subsection we assume that

- (i) *there are needles with only two orientations, and*
- (ii) *needles of the same orientation are of the same length but needles along different orientations could be of different lengths.*

Without loss of generality we assume that needles are either horizontal or at an angle $\alpha \in (0, \pi]$. Needles which are horizontal are of length R_0 and needles at an angle α are of length R_α . The probability that a randomly chosen needle is horizontal is p and that it is at an angle α is $1 - p$.

In this case $\Lambda(k, \ell)$ is the set of clusters containing k horizontal needles and ℓ needles at an angle α with respect to the x -axis. We show that

Theorem 2.1 *Let $m = k + \ell$, $k, \ell \geq 1$, $\alpha \in (0, \pi)$ and $0 < R_0, R_\alpha$. As $\lambda \rightarrow \infty$, we have*

$$(i) \quad \mu_{\lambda\rho}(C_0 \in \Lambda(k, \ell) \mid \Gamma_0)$$

$$\sim \left(\frac{1}{\lambda |B_{R_0, R_\alpha}^{0, \alpha}|} \right)^{m-3} e^{-\lambda |B_{R_0, R_\alpha}^{0, \alpha}|} (pq)^{-2(m-1)} mp^{3k} k! q^{3\ell} \ell!,$$

where $a(\lambda) \sim b(\lambda)$ means that $\frac{a(\lambda)}{b(\lambda)} \rightarrow 1$ as $\lambda \rightarrow \infty$;

$$(ii) \quad p_{\lambda, m}(k, \ell) := \mu_{\lambda\rho}(\#C_0 = (k, \ell) \mid \#C_0 = (k', \ell'), \quad k' + \ell' = m)$$

$$\sim \frac{p^{3k} k! q^{3\ell} \ell!}{\sum_{k+\ell=m} p^{3k} k! q^{3\ell} \ell!}.$$

From the proof of the above theorem we also observe:

Remark: (a) The centres of the needles at an angle α comprising the cluster C_0 lie in a region whose area is of the order $o(\lambda^{-1+\delta})$ for any $\delta > 0$ as $\lambda \rightarrow \infty$ (see Figure 1). This is the phenomenon of compression/rarefaction as observed by Alexander (1993) and Sarkar (1998) in the case of high intensity Boolean models with balls as the underlying shapes.

(b) Moreover, this region is uniformly distributed in the parallelogram $B_{R_0, R_\alpha}^{0, \alpha}$.

An interesting observation from (ii) above is that asymptotically, as $\lambda \rightarrow \infty$, the conditional probability $p_{\lambda, m}(k, \ell)$ of the needles comprising the *finite* cluster C_0 , is independent of both the angle α as well as R_0 and R_α , the lengths of the needles. This is not surprising because the model is invariant under affine transformations. Now let $p_m(k, \ell) := \lim_{\lambda \rightarrow \infty} p_{\lambda, m}(k, \ell)$. We also observe from Theorem 2.1 (ii) that, as $m \rightarrow \infty$,

$$\begin{aligned} p_m(m-1, 1) &\rightarrow 1 && \text{for } p > q, \\ p_m(1, m-1) &\rightarrow 1 && \text{for } p < q, \\ p_m(1, m-1) = p_m(m-1, 1) &\rightarrow \frac{1}{2} && \text{for } p = q. \end{aligned}$$

Moreover, let k and m both approach infinity in such a way that $(k/m) \rightarrow s$, for some $s \in [0, 1]$, then we have

$$\lim_{\substack{m \rightarrow \infty \\ (k/m) \rightarrow s}} \frac{1}{m} \log p_m(k, \ell) = H(s), \quad (2.4)$$

where

$$H(s) = s \log s + (1-s) \log(1-s) + \begin{cases} 3(1-s) \log(q/p), & \text{if } p > q, \\ 3s \log(p/q), & \text{if } p < q, \\ 0, & \text{if } p = q, \end{cases}$$

from which we may deduce that as $m \rightarrow \infty$, for $0 \leq a \leq b \leq 1$,

$P(\text{the proportion } (k/m) \text{ of horizontal needles in the cluster lies between } a \text{ and } b)$

$\sim \exp\{\sup_{s \in (a, b)} H(s)\}.$

2.3 Needles of three types

In this subsection we assume that

- (i) *there are needles with only three orientations – 0, α and β ,*
- (ii) *needles of the same orientation are of the same length.*

Here the results are significantly different from those obtained in the previous section. In particular the absence of any affine invariance leads to the dependence of the results on both the length and orientation of the needles through the following quantities

$$H_\alpha = \frac{R_\alpha}{\sin \beta}, \quad H_\beta = \frac{R_\beta}{\sin \alpha}, \quad H_0 = \frac{R_0}{\sin(\beta - \alpha)}. \quad (2.5)$$

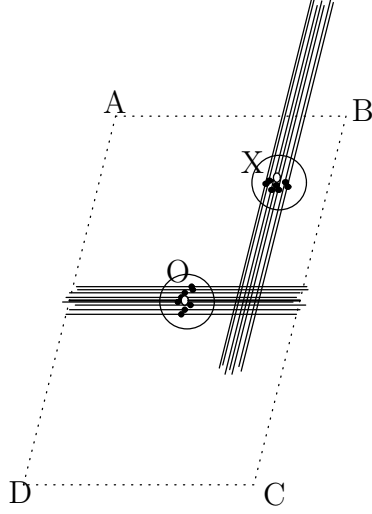


Figure 1: The finite cluster for large λ . The region X which contains the centres of the needles at an angle α w.r.t. the x -axis is uniformly distributed in the parallelogram $ABCD$.

By a suitable scaling we take

$$H_0 = 1 \text{ and let } H_\alpha = a, \ H_\beta = b \text{ after the scaling.} \quad (2.6)$$

As the following theorem exhibits, the asymptotic (as $\lambda \rightarrow \infty$) composition of the finite cluster contains needles of only two distinct orientation, while the third does not figure at all.

Here we use the shorthand “ $A(x, y)$ occurs” to mean that as $\lambda \rightarrow \infty$ the asymptotic shape of C_0 consists of needles only in the directions x and y . Moreover, as in Remark after Theorem 2.1, the centres of the surviving needles in a particular orientation has area of the order $o(\lambda^{-1+\delta})$ for any $\delta > 0$ as $\lambda \rightarrow \infty$, and is uniformly distributed in a region which depends on the parameters of the model. In certain cases when needles in two directions are of the same length and different from the length of the needle in the third direction, then, depending on the other parameters of the model, i.e. p_0, p_α and p_β , the area of this region where the centres of the surviving needles lie shrink to zero, and in this case we say that “fixation occurs”.

Theorem 2.2 *Given that C_0 consists of m needles,*

(1) *for $a, b \geq 2$;*

- (i) *if $(ab - a + 1/4)p_\beta + a < (ab - b + 1/4)p_\alpha + b$, then $A(0, \alpha)$ occurs,*
- (ii) *if $(ab - a + 1/4)p_\beta + a > (ab - b + 1/4)p_\alpha + b$, then $A(0, \beta)$ occurs, and*

- (iii) if $(ab - a + 1/4)p_\beta + a = (ab - b + 1/4)p_\alpha + b$, then both $A(0, \alpha)$ and $A(0, \beta)$ have positive probabilities of occurrence;
- (2) for $1/2 < \min\{a, b\} < 2$ and $a \neq b$, $a, b \neq 1$ and for $x, y, z \in \{0, \alpha, \beta\}$ let

$$f(x, y, z) := p_x H_x \max\{H_y, H_z\} + p_x \min\{H_y, H_z\}^2/4 + (1 - p_x)H_y H_z,$$
 - (i) $A(\alpha, \beta)$ occurs when $f(0, \alpha, \beta) < \min\{f(\beta, 0, \alpha), f(\alpha, \beta, 0)\}$
 - (ii) $A(0, \alpha)$ and $A(0, \beta)$ have positive probabilities of occurrence, when $f(\beta, 0, \alpha) = f(\alpha, \beta, 0) < f(0, \alpha, \beta)$, and
 - (iii) $A(\alpha, \beta)$, $A(0, \alpha)$ and $A(0, \beta)$ all have positive probabilities of occurrence when $f(\beta, 0, \alpha) = f(\alpha, \beta, 0) = f(0, \alpha, \beta)$;
- (3) for $0 < a = b < 1$, and,
 - (i) for $p_0 \leq \min\{p_\alpha, p_\beta\}$, $A(\alpha, \beta)$ occurs,
 - (ii) for $p_0 > \min\{p_\alpha, p_\beta\}$,
 - if $a < \mathbf{I}_1(p_0, p_\alpha, p_\beta) := 1 - \frac{p_0 - \min\{p_\alpha, p_\beta\}}{4 - 3p_0 - \min\{p_\alpha, p_\beta\}}$, then $A(\alpha, \beta)$ and fixation occurs, while,
 - if $a \geq \mathbf{I}_1(p_0, p_\alpha, p_\beta)$, $A(0, \alpha)$ occurs for $p_\alpha > p_\beta$ and both $A(0, \alpha)$ and $A(0, \beta)$ have positive probability of occurrence for $p_\alpha = p_\beta$;
- (4) for $1 < a = b < 2$, and,
 - (i) for $p_0 < \min\{p_\alpha, p_\beta\}$,
 - if $a < \mathbf{I}_2(p_0, p_\alpha, p_\beta) := \frac{2 \max\{p_\alpha, p_\beta\} + \sqrt{4 \max\{p_\alpha, p_\beta\}^2 + 4 p_\alpha p_\beta + p_0 \min\{p_\alpha, p_\beta\}}}{4 \max\{p_\alpha, p_\beta\} + p_0}$, then $A(\alpha, \beta)$ and fixation occurs, while,
 - if $a \geq \mathbf{I}_2(p_0, p_\alpha, p_\beta)$, $A(0, \alpha)$ occurs for $p_\alpha > p_\beta$ and both $A(0, \alpha)$ and $A(0, \beta)$ have positive probability of occurrence for $p_\alpha = p_\beta$,
 - (ii) for $\min\{p_\alpha, p_\beta\} \leq p_0$, $A(0, \alpha)$ occurs for $p_\alpha > p_\beta$ and both $A(0, \alpha)$ and $A(0, \beta)$ have positive probability of occurrence for $p_\alpha = p_\beta$;
- (5) for $a = b = 1$, fixation always occurs and
 - (i) $A(x, y)$ occurs when $p_z < \min\{p_x, p_y\}$,
 - (ii) with equal probability $A(x, y)$ and $A(x, z)$ occur when $p_y = p_z < p_x$, and
 - (iii) with equal probability $A(x, y)$, $A(y, z)$ and $A(z, x)$ occur when $p_x = p_y = p_z$;

Observe that for $\min a, b \leq 1/2$:

- (A) If $b, 1 \geq 2a$, then by the scaling which transforms a to 1, b to b/a and 1 to $1/a$, the resulting asymptotic cluster may be read from (1) of Theorem 2.2. Similarly if $a, 1 \geq 2b$, we may scale suitably to obtain a situation as in (1) of Theorem 2.2.

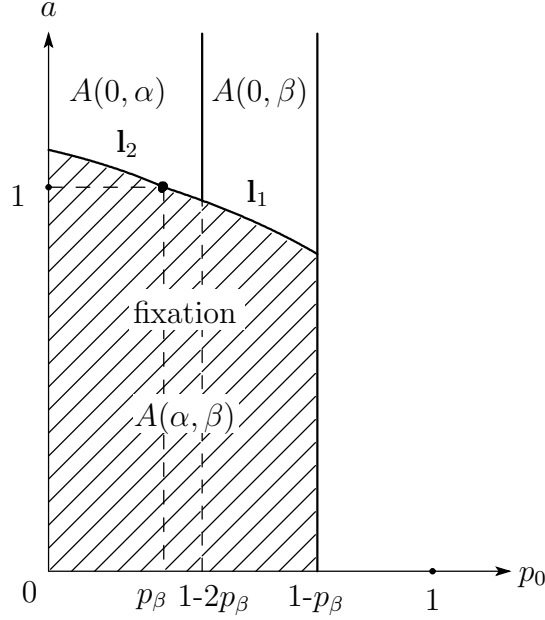


Figure 2: The diagram in the case that $a = b$ and $p_\beta \in (0, 1/3)$. The curved line is the line $\mathbf{l}_1 1_{\{0 \leq l_1 \leq 1\}} + \mathbf{l}_2 1_{\{1 \leq l_1 \leq 2\}}$. For $p_0 > 0$ and a below this line $A(\alpha, \beta)$ occurs, while for a above the line $A(0, \beta)$ occurs when $p_\alpha < p_\beta$. At $p_0 = 0$, only $A(\alpha, \beta)$ occurs.

(B) If either $a/2 < \min\{1, b\} < 2a, a \neq b, a, b \neq 1$, or $b/2 < \min\{1, a\} < 2b, a \neq b, a, b \neq 1$, then scaling shows that (2) of Theorem 2.2 may be used to yield the asymptotic shape.

(C) If either $0 < b = 1 < a$ or $0 < a = 1 < b$, then scaling shows that (3) of Theorem 2.2 may be used to yield the asymptotic shape.

(D) If either $a < b = 1 < 2a$ or $b < a = 1 < 2b$, then scaling shows that (4) of Theorem 2.2 may be used to yield the asymptotic shape.

Thus the above four observations demonstrate that Theorem 2.2 yields the asymptotic shapes for all possible values of a and b .

To prove the above theorem we need to know the conditional probability of the composition of a cluster given that it is finite.

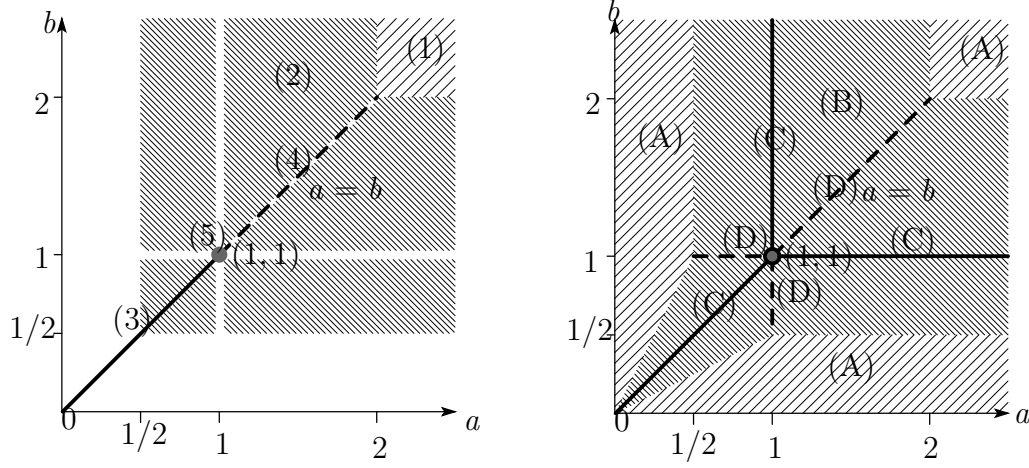


Figure 3: The various regions where Theorem the various parts of Theorem 2.2 hold.

3 Proof of Theorem 2.1

3.1 General set-up

For $\mathbf{k} \in (\mathbb{N} \cup 0)^d$, $d \geq 2$, with $|\mathbf{k}| = m$, let $\Lambda(\mathbf{k})$ and Γ_0 be as in Section 2.1. First we calculate $\mu_{\lambda\rho}(C_0 \in \Lambda(\mathbf{k}) | \Gamma_0)$. Suppose that $w_m = (\mathbf{0}, \alpha_{j_0}, R_{j_0})$ for some $j_0 \in \{1, 2, \dots, d\}$. We have

$$\begin{aligned} & \mu_{\lambda\rho}(C_0 \in \Lambda(\mathbf{k}) \mid w_m \in \xi) \\ &= \int_{\mathcal{M}} \mu_{\lambda\rho}(d\xi) \sum_{\{\mathbf{w}_{m-1}\} \subset \xi} 1_{\Lambda(\mathbf{k})}(C_0(\mathbf{w}_m)) 1_{\{S(\xi \setminus \{\mathbf{w}_m\}) \cap S(\{\mathbf{w}_m\}) = \emptyset\}}, \end{aligned}$$

where \mathbf{w}_m , $\{\mathbf{w}_m\}$ and $C_0(\mathbf{w}_m)$ are as defined in (2.3). Thus,

$$\begin{aligned} & \mu_{\lambda\rho}(C_0 \in \Lambda(\mathbf{k}) \mid w_m \in \xi) \\ &= \frac{\lambda^{m-1}}{(m-1)!} \int_{\mathcal{M}} \mu_{\lambda\rho}(d\eta) \int_{\mathcal{R}^{m-1}} \rho^{\otimes(m-1)}(d\mathbf{w}_{m-1}) 1_{\Lambda(\mathbf{k})}(C_0(\mathbf{w}_m)) 1_{\{S(\eta) \cap S(\{\mathbf{w}_m\}) = \emptyset\}} \\ &= \frac{\lambda^{m-1}}{(m-1)!} \int_{\mathcal{R}^{m-1}} \rho^{\otimes(m-1)}(d\mathbf{w}_{m-1}) 1_{\Lambda(\mathbf{k})}(C_0(\mathbf{w}_m)) e^{-\lambda\rho(w:S(w) \cap S(\{\mathbf{w}_m\}) \neq \emptyset)}. \end{aligned}$$

Note that $S(x, \theta, r) \cap S(\{\mathbf{w}_m\}) \neq \emptyset$ if and only if $x \in \cup_{i=1}^m B_{r_i, r}^{\theta_i, \theta}(x_i)$ where $w_i = (x_i, \theta_i, r_i)$, $i = 1, 2, \dots, m$. Hence,

$$\rho(w : S(w) \cap S(\{\mathbf{w}_m\}) \neq \emptyset) = \sum_{j=1}^d p_j \left| \bigcup_{i=1}^m B_{r_i, R_j}^{\theta_i, \alpha_j}(x_i) \right|,$$

and so

$$\begin{aligned} \mu_{\lambda\rho}(C_0 \in \Lambda(\mathbf{k}) \mid w_m \in \xi) &= \frac{\lambda^{m-1}}{(m-1)!} \int_{\mathcal{R}^{m-1}} \rho^{\otimes(m-1)}(d\mathbf{w}_{m-1}) 1_{\Lambda(\mathbf{k})}(C_0(\mathbf{w}_m)) \\ &\quad \times \exp \left[-\lambda \sum_{j=1}^d p_j \left| \bigcup_{i=1}^m B_{r_i, R_j}^{\theta_i, \alpha_j}(x_i) \right| \right]. \end{aligned}$$

Let

$$\begin{aligned} F_{\lambda}^{\alpha_{j_0}}(\mathbf{k}) &= \int_{(\mathbb{R}^2)^{k_1}} d\mathbf{x}_{1,k_1} \int_{(\mathbb{R}^2)^{k_2}} d\mathbf{x}_{2,k_2} \cdots \int_{(\mathbb{R}^2)^{k_{j_0}-1}} d\mathbf{x}_{j_0, k_{j_0}-1} \cdots \int_{(\mathbb{R}^2)^{k_d}} d\mathbf{x}_{d, k_d} \\ &\quad \times 1_{\Lambda(\mathbf{k})}(C_0(\mathbf{x})) \exp \left[-\lambda \sum_{j=1}^d p_j \left| \bigcup_{i=1, k_i \neq 0}^d B_{R_i, R_j}^{\alpha_i, \alpha_j}(\mathbf{x}_{i, k_i}) \right| \right], \end{aligned}$$

where $C_0(\mathbf{x}) = C_0(\mathbf{x}_{1,k_1}, \mathbf{x}_{2,k_2}, \dots, \mathbf{x}_{d,k_d}) = C_0(\bigcup_{j=1}^d \{(x_{j,i}, \alpha_j, R_j) : i = 1, \dots, k_j\})$. From the translation invariance of Lebesgue measure it is obvious that if $k_j, k_{j'} \geq 1$, then $F_{\lambda}^{\alpha_j}(\mathbf{k}) = F_{\lambda}^{\alpha_{j'}}(\mathbf{k})$. Thus writing $F_{\lambda}(\mathbf{k})$ for $F_{\lambda}^{\alpha_j}(\mathbf{k})$, since $\mu_{\lambda\rho}((0, \alpha_j, R_j) \in \xi \mid \Gamma_0) = p_j$, we have

$$\mu_{\lambda\rho}(C_0 \in \Lambda(\mathbf{k}) \mid \Gamma_0) = \frac{\lambda^{m-1} m!}{(m-1)!} \prod_{j=1}^d \frac{p_j^{k_j}}{k_j!} F_{\lambda}(\mathbf{k}) = \lambda^{|\mathbf{k}|-1} |\mathbf{k}| \prod_{j=1}^d \frac{p_j^{k_j}}{k_j!} F_{\lambda}(\mathbf{k}). \quad (3.1)$$

3.2 Proof of Theorem 2.1

To prove Theorem 2.1, observe first that in the case when we have needles with only two orientations, the Radon measure ρ is given by

$$\rho(dx \, d\theta \, dr) = dx \{p\delta_0(d\theta)\delta_{R_0}(dr) + q\delta_{\alpha}(d\theta)\delta_{R_{\alpha}}(dr)\}, \quad (3.2)$$

where $q = 1 - p$.

Also, the Poisson point process being invariant under a measure-preserving affine transformation, we may assume that $\alpha = \pi/2$.

From (3.1) we have

$$\begin{aligned} \mu_{\lambda\rho}(C_0 \in \Lambda(k, \ell) \mid \Gamma_0) &= \lambda^{k+\ell-1} (k+\ell) \frac{p^k q^{\ell}}{k! \ell!} F_{\lambda}^0((k, \ell)) \\ &= \lambda^{k+\ell-1} (k+\ell) \frac{p^k q^{\ell}}{k! \ell!} e^{-\lambda |B_{R_0, R_{\alpha}}^{0, \alpha}|} f_{\lambda}(k, \ell), \end{aligned}$$

with

$$f_{\lambda}(k, \ell) := \int_{(\mathbb{R}^2)^{k-1}} d\mathbf{x}_{k-1} \int_{(\mathbb{R}^2)^{\ell}} d\mathbf{y}_{\ell} 1_{\Lambda(k, \ell)}(C_0(\mathbf{x}_k, \mathbf{y}_{\ell})) \chi_{p\lambda}^{0, \alpha}(\mathbf{y}_{\ell}) \chi_{q\lambda}^{0, \alpha}(\mathbf{x}_k),$$

$$\chi_c^{\theta_1, \theta_2}(\mathbf{x}) = \exp \left[-c \{ |B_{R_{\theta_1}, R_{\theta_2}}^{\theta_1, \theta_2}(\mathbf{x})| - |B_{R_{\theta_1}, R_{\theta_2}}^{\theta_1, \theta_2}| \} \right] \quad (3.3)$$

(note here that $x_k = \mathbf{0}$). Now consider the event $A(\mathbf{x}_k, \mathbf{y}_\ell, k, \ell) := \{C_0 \text{ contains exactly } m \text{ needles } (\mathbf{0}, 0, 1/2), (x_1, 0, 1/2), \dots, (x_{k-1}, 0, 1/2), (y_1, \frac{\pi}{2}, 1/2), \dots, (y_\ell, \frac{\pi}{2}, 1/2)\}$. By the affine invariance of the Lebesgue measure

$$\begin{aligned} f_\lambda(k, \ell) &= |B_{R_0, R_\alpha}^{0, \alpha}|^{m-1} \int_{(\mathbb{R}^2)^{k-1}} d\mathbf{x}_{k-1} \int_{(\mathbb{R}^2)^\ell} d\mathbf{y}_\ell 1_{A(\mathbf{x}_k, \mathbf{y}_\ell, k, \ell)} \\ &\quad \times \exp[-\lambda p |B_{R_0, R_\alpha}^{0, \alpha}| \{|B_{\frac{1}{2}}(\mathbf{y}_\ell)| - |B_{\frac{1}{2}}|\}] \\ &\quad \times \exp[-\lambda q |B_{R_0, R_\alpha}^{0, \alpha}| \{|B_{\frac{1}{2}}(\mathbf{x}_k)| - |B_{\frac{1}{2}}|\}], \end{aligned} \quad (3.4)$$

where $B_R = [-R, R]^2$, $B_R(x) = B_R + x$ and $B_R(\mathbf{x}_k) = \cup_{i=1}^k B_R(x_i)$.

For the proof of Theorem 2.1 we need to obtain lower and upper bounds of $f_\lambda(k, l)$ which we later show to agree as $\lambda \rightarrow \infty$. To this end we need the following lemma whose proof is given in the appendix. We put

$$M(\mathbf{u}_k) = \max_{1 \leq i, j \leq k} |u_i - u_j|, \quad \mathbf{u}_k = (u_1, u_2, \dots, u_k) \in (\mathbb{R})^k.$$

and $C_{\alpha, \beta} = \sin \alpha \sin \beta \sin(\alpha - \beta)$. The quantities h_α , h_β and h_0 are as defined in Section 2.1.

Lemma 3.1 *Let $\mathbf{x}_k = (x_1, x_2, \dots, x_k) \in (\mathbb{R}^2)^k$, $x_i = (x_i(1), x_i(2))$ with $x_k = \mathbf{0}$. Also let $\mathbf{x}_k^j = (x_1(j), x_2(j), \dots, x_k(j)) \in (\mathbb{R})^k$, $j = 1, 2$. We have*

$$\begin{aligned} |B_{R_0, R_{\pi/2}}^{0, \pi/2}(\mathbf{x}_k) \setminus B_{R_0, R_{\pi/2}}^{0, \pi/2}| &\leq 2R_0 M(\mathbf{x}_k^2) + 2R_{\pi/2} M(\mathbf{x}_k^1) \\ &\quad + M(\mathbf{x}_k^1) M(\mathbf{x}_k^2), \end{aligned} \quad (3.5)$$

and, if $B_{R_0, R_{\pi/2}}^{0, \pi/2}(\mathbf{x}_k)$ is connected, then we have

$$|B_{R_0, R_{\pi/2}}^{0, \pi/2}(\mathbf{x}_k) \setminus B_{R_0, R_{\pi/2}}^{0, \pi/2}| \geq R_0 M(\mathbf{x}_k^2) + R_{\pi/2} M(\mathbf{x}_k^1), \quad (3.6)$$

$$\begin{aligned} |B_{R_0, R_{\pi/2}}^{0, \pi/2}(\mathbf{x}_k) \setminus B_{R_0, R_{\pi/2}}^{0, \pi/2}| &\geq 2R_0 M(\mathbf{x}_k^2) + 2R_{\pi/2} M(\mathbf{x}_k^1) \\ &\quad - M(\mathbf{x}_k^1) M(\mathbf{x}_k^2). \end{aligned} \quad (3.7)$$

More generally, in the bases e_α, e_β , for $\alpha, \beta \in (0, \pi)$, we have

$$\begin{aligned} |B_{R_\alpha, R_\beta}^{\alpha, \beta}(\mathbf{x}_k) \setminus B_{R_\alpha, R_\beta}^{\alpha, \beta}| &\leq 2C_{\alpha, \beta} \{H_\alpha M(h_\beta(\mathbf{x}_k)) + H_\beta M(h_\alpha(\mathbf{x}_k))\} \\ &\quad + C_{\alpha, \beta} M(h_\beta(\mathbf{x}_k)) M(h_\alpha(\mathbf{x}_k)), \end{aligned} \quad (3.8)$$

and, if $B_{R_\alpha, R_\beta}^{\alpha, \beta}(\mathbf{x}_k)$ is connected, then we have

$$|B_{R_\alpha, R_\beta}^{\alpha, \beta}(\mathbf{x}_k) \setminus B_{R_\alpha, R_\beta}^{\alpha, \beta}| \geq C_{\alpha, \beta} \{H_\alpha M(h_\beta(\mathbf{x}_k)) + H_\beta M(h_\alpha(\mathbf{x}_k))\}, \quad (3.9)$$

$$\begin{aligned} |B_{R_\alpha, R_\beta}^{\alpha, \beta}(\mathbf{x}_k) \setminus B_{R_\alpha, R_\beta}^{\alpha, \beta}| &\geq 2C_{\alpha, \beta} \{H_\alpha M(h_\beta(\mathbf{x}_k)) + H_\beta M(h_\alpha(\mathbf{x}_k))\} \\ &\quad - C_{\alpha, \beta} M(h_\beta(\mathbf{x}_k)) M(h_\alpha(\mathbf{x}_k)). \end{aligned} \quad (3.10)$$

Now we evaluate the bounds of $f_\lambda(k, \ell)$.

LOWER BOUND : By (3.5) of Lemma 3.1, taking $x_k = \mathbf{0}$ we have

$$\begin{aligned}
f_\lambda(k, \ell) &\geq |B_{R_0, R_{\pi/2}}^{0, \pi/2}|^{m-1} \int_{(\mathbb{R}^2)^{k-1}} d\mathbf{x}_{k-1} \int_{(\mathbb{R}^2)^\ell} d\mathbf{y}_\ell \mathbf{1}_{A(\mathbf{x}_k, \mathbf{y}_\ell, k, \ell)} \\
&\quad \times \exp[-\lambda q |B_{R_0, R_{\pi/2}}^{0, \pi/2}| (M(\mathbf{x}_k^1) + M(\mathbf{x}_k^2))] \\
&\quad \times \exp[-\lambda p |B_{R_0, R_{\pi/2}}^{0, \pi/2}| (M(\mathbf{y}_\ell^1) + M(\mathbf{y}_\ell^2))] \\
&\quad \times \exp[-\lambda |B_{R_0, R_{\pi/2}}^{0, \pi/2}| \{qM(\mathbf{x}_k^1)M(\mathbf{x}_k^2) + pM(\mathbf{y}_\ell^1)M(\mathbf{y}_\ell^2)\}]. \tag{3.11}
\end{aligned}$$

Let $L(\lambda)$ be such that, as $\lambda \rightarrow \infty$, $\lambda L(\lambda) \rightarrow \infty$ and $\lambda(L(\lambda))^2 \rightarrow 0$. For $x_k = \mathbf{0}$, if $\{x_i, 1 \leq i \leq k-1\} \subset B_{L(\lambda)}$, $\{y_i, 1 \leq i \leq \ell-1\} \subset B_{L(\lambda)}(y_\ell)$, and $y_\ell \in B_{1/2-L(\lambda)}$, then, for λ sufficiently large, we have that $A(\mathbf{x}_k, \mathbf{y}_\ell, k, \ell)$ occurs, and the expression on the right of the inequality (3.11) is bounded from below by

$$\begin{aligned}
&|B_{R_0, R_{\pi/2}}^{0, \pi/2}|^{m-1} \int_{(B_{L(\lambda)})^{k-1}} d\mathbf{x}_{k-1} \int_{B_{1/2-L(\lambda)}} dy_\ell \int_{(B_{L(\lambda)}(y_\ell))^{\ell-1}} d\mathbf{y}_{\ell-1} \\
&\quad \times \exp[-\lambda q |B_{R_0, R_{\pi/2}}^{0, \pi/2}| (M(\mathbf{x}_k^1) + M(\mathbf{x}_k^2))] \\
&\quad \times \exp[-\lambda p |B_{R_0, R_{\pi/2}}^{0, \pi/2}| (M(\mathbf{y}_k^1) + M(\mathbf{y}_k^2))] \\
&\quad \times \exp[-\lambda |B_{R_0, R_{\pi/2}}^{0, \pi/2}| \{qM(\mathbf{x}_k^1)M(\mathbf{x}_k^2) + pM(\mathbf{y}_k^1)M(\mathbf{y}_k^2)\}] \\
&\geq |B_{R_0, R_{\pi/2}}^{0, \pi/2}|^{m-1} e^{-16R_0 R_{\pi/2}(p+q)\lambda(L(\lambda))^2} |B_{1/2-L(\lambda)}| \int_{(B_{L(\lambda)})^{k-1}} d\mathbf{x}_{k-1} \int_{(B_{L(\lambda)})^{\ell-1}} d\mathbf{y}_{\ell-1} \\
&\quad \times \exp[-\lambda q |B_{R_0, R_{\pi/2}}^{0, \pi/2}| (M(\mathbf{x}_k^1) + M(\mathbf{x}_k^2))] \\
&\quad \times \exp[-\lambda p |B_{R_0, R_{\pi/2}}^{0, \pi/2}| (M(\mathbf{y}_k^1) + M(\mathbf{y}_k^2))] \\
&= e^{-16R_0 R_{\pi/2}\lambda(L(\lambda))^2} |B_{1/2-L(\lambda)}| (q\lambda)^{-2(k-1)} (p\lambda)^{-2(\ell-1)} |B_{R_0, R_{\pi/2}}^{0, \pi/2}|^{-(m-3)} \\
&\quad \times \int_{(B_{4R_0 R_{\pi/2} q \lambda L(\lambda)})^{k-1}} d\mathbf{u}_{k-1} \exp[-M(\mathbf{u}_k^1) - M(\mathbf{u}_k^2)] \\
&\quad \times \int_{(B_{4R_0 R_{\pi/2} p \lambda L(\lambda)})^{\ell-1}} d\mathbf{v}_{\ell-1} \exp[-M(\mathbf{v}_k^1) - M(\mathbf{v}_k^2)] \tag{3.12}
\end{aligned}$$

where $\mathbf{u}_k = (u_1, \dots, u_k)$ and $\mathbf{v}_\ell = (v_1, \dots, v_\ell)$ with $v_\ell = u_k = \mathbf{0}$. Thus we have

$$\begin{aligned}
f_\lambda(k, \ell) &\geq e^{-16R_0 R_{\pi/2} \lambda (L(\lambda))^2} |B_{R-L(\lambda)}| \lambda^{-2(m-2)} |B_{R_0, R_{\pi/2}}^{0, \pi/2}|^{-(m-3)} q^{-2(k-1)} p^{-2(\ell-1)} \\
&\times \left[\int_{-4R_0 R_{\pi/2} q \lambda L(\lambda)}^{4R_0 R_{\pi/2} q \lambda L(\lambda)} da_1 \cdots \int_{-4R_0 R_{\pi/2} q \lambda L(\lambda)}^{4R_0 R_{\pi/2} q \lambda L(\lambda)} da_{k-1} \exp\left\{-\max_{1 \leq i, j \leq k} |a_i - a_j|\right\} \right]^2 \\
&\times \left[\int_{-4R_0 R_{\pi/2} p \lambda L(\lambda)}^{4R_0 R_{\pi/2} p \lambda L(\lambda)} db_1 \cdots \int_{-4R_0 R_{\pi/2} p \lambda L(\lambda)}^{4R_0 R_{\pi/2} p \lambda L(\lambda)} db_{\ell-1} \exp\left\{-\max_{1 \leq i, j \leq \ell} |b_i - b_j|\right\} \right]^2 \quad (3.13)
\end{aligned}$$

where $a_k = b_\ell = 0$.

Since $e^{-16R_0 R_{\pi/2} \lambda (L(\lambda))^2} = 1 - O(\lambda(L(\lambda))^2)$ as $\lambda \rightarrow 0$, by (3.13) and the above lemma we obtain that, as $\lambda \rightarrow 0$,

$$f_\lambda(k, \ell) \geq \left[\left(\frac{1}{\lambda} \right)^{2(m-2)} \left(\frac{1}{|B_{R_0, R_{\pi/2}}^{0, \pi/2}|} \right)^{m-3} q^{-2(k-1)} p^{-2(\ell-1)} (k!)^2 (\ell!)^2 \right] (1 - O(\lambda(L(\lambda))^2)). \quad (3.14)$$

Now we will obtain the upper bound of $f_\lambda(k, \ell)$.

UPPER BOUND: For $L(\lambda)$ as earlier, consider the event

$$E := \{x_1, \dots, x_{k-1} \in B_{L(\lambda)}, y_1, \dots, y_{\ell-1} \in B_{L(\lambda)}(y_\ell)\}.$$

If $x_k = \mathbf{0}$, for $E \cap A(\mathbf{x}_k, \mathbf{y}_\ell, k, \ell)$ to occur, we must have $y_\ell \in B_{(1/2)+L(\lambda)}$. Thus from (3.4) we have

$$\begin{aligned}
f_\lambda(k, \ell) &\leq |B_{R_0, R_{\pi/2}}^{0, \pi/2}|^{m-1} \int_{(\mathbb{R}^2)^{k-1}} d\mathbf{x}_{k-1} \int_{\mathbb{R}^2} dy_\ell \int_{(\mathbb{R}^2)^{\ell-1}} d\mathbf{y}_{\ell-1} \\
&\times (1_{E \cap \{y_\ell \in B_{(1/2)+L(\lambda)}\}} + 1_{E^c} 1_{A(\mathbf{x}_k, \mathbf{y}_\ell, k, \ell)}) \\
&\times \exp[-\lambda p |B_{R_0, R_{\pi/2}}^{0, \pi/2}| \{|B_{\frac{1}{2}}(\mathbf{y}_\ell)| - |B_{\frac{1}{2}}|\}] \\
&\times \exp[-\lambda q |B_{R_0, R_{\pi/2}}^{0, \pi/2}| \{|B_{\frac{1}{2}}(\mathbf{x}_k)| - |B_{\frac{1}{2}}|\}]. \quad (3.15)
\end{aligned}$$

On opening the parenthesis $(1_{E \cap \{y_\ell \in B_{(1/2)+L(\lambda)}\}} + 1_{E^c} 1_{A(\mathbf{x}_k, \mathbf{y}_\ell, k, \ell)})$ in the expression on the right of the inequality (3.15) above the term involving $1_{E \cap \{y_\ell \in B_{(1/2)+L(\lambda)}\}}$, for large λ , may be bounded from above by

$$\begin{aligned}
&e^{4\lambda(L(\lambda))^2} |B_{1/2+L(\lambda)}| (q\lambda)^{-2(k-1)} (p\lambda)^{-2(\ell-1)} |B_{R_0, R_{\pi/2}}^{0, \pi/2}|^{-(m-3)} \\
&\times \int_{(B_{4R_0 R_{\pi/2} q \lambda L(\lambda)})^{k-1}} d\mathbf{u}_{k-1} \exp[-M(\mathbf{u}_k^1) - M(\mathbf{u}_k^2)] \\
&\times \int_{(B_{4R_0 R_{\pi/2} p \lambda L(\lambda)})^{\ell-1}} d\mathbf{v}_{\ell-1} \exp[-M(\mathbf{v}_\ell^1) - M(\mathbf{v}_\ell^2)]. \quad (3.16)
\end{aligned}$$

(Here we have used the inequality (3.7) of Lemma 3.1 and calculations similar to those leading to (3.12).)

Using the inequality (3.6) of Lemma 3.1 we bound the expression involving $1_{E^c} 1_{A(\mathbf{x}_k, \mathbf{y}_\ell, k, \ell)}$ in the right of the inequality (3.15) by $|B_{R_0, R_{\pi/2}}^{0, \pi/2}|^{m-1} \{I_1 + I_2\}$, where

$$I_1 = \int_{(\mathbb{R}^2)^{k-1} \setminus (B_{L(\lambda)})^{k-1}} d\mathbf{x}_{k-1} \int_{B_m} dy_\ell \int_{(\mathbb{R}^2)^{\ell-1}} d\mathbf{y}_{\ell-1} \\ \times \exp\{-(q/2)\lambda(M(\mathbf{x}_k^1) + M(\mathbf{x}_k^2))\} \exp\{-(p/2)\lambda(M(\mathbf{y}_\ell^1) + M(\mathbf{y}_\ell^2))\}$$

and

$$I_2 = \int_{(\mathbb{R}^2)^{k-1}} d\mathbf{x}_{k-1} \int_{B_m} dy_\ell \int_{(\mathbb{R}^2)^{\ell-1} \setminus (B_{L(\lambda)})^{\ell-1}} d\mathbf{y}_{\ell-1} \\ \times \exp\{-(q/2)\lambda(M(\mathbf{x}_k^1) + M(\mathbf{x}_k^2))\} \exp\{-(p/2)\lambda(M(\mathbf{y}_\ell^1) + M(\mathbf{y}_\ell^2))\}.$$

(Here we note that for the m needles to be connected, y_ℓ must be in a box of sides of length m centred at $x_k = \mathbf{0}$.)

Taking $a_k = 0$, it is easy to see that

$$\int_{\mathbb{R}^{k-1}} da_1 \cdots da_{k-1} \exp\left\{-\max_{1 \leq i, j \leq k} |a_i - a_j|\right\} = k!.$$

Using this equation and calculations as in (3.13) and (3.14), for $\lambda \rightarrow \infty$, the expression in (3.16) may be bounded above by

$$\left[\left(\frac{1}{\lambda}\right)^{2(m-2)} \left(\frac{1}{|B_{R_0, R_{\pi/2}}^{0, \pi/2}|}\right)^{m-3} q^{-2(k-1)} p^{-2(\ell-1)} (k!)^2 (\ell!)^2 \right] (1 + O(\lambda(L(\lambda))^2)).$$

Thus to show that, asymptotically in λ the lower bound (3.14) of $f(k, \ell)$ agrees with its upper bound it suffices to show that

$$I_1 + I_2 = O(\lambda^{-2m-3}) \text{ as } \lambda \rightarrow \infty. \quad (3.17)$$

To estimate the integrals I_1 and I_2 , we use the symmetry of the integrand in I_1 to obtain

$$I_1 \leq 4(k-1) \int_{(\mathbb{R}^2)^{k-2}} d\mathbf{x}_{k-2} \int_{\mathbb{R}} dx_{k-1}^1 \int_{L(\lambda)}^\infty dx_{k-1}^2 |B_m| \int_{(\mathbb{R}^2)^{\ell-1}} d\mathbf{y}_{\ell-1} \\ \times \exp\{-(q/2)\lambda(M(\mathbf{x}_k^1) + M(\mathbf{x}_k^2))\} \exp\{-(p/2)\lambda(M(\mathbf{y}_\ell^1) + M(\mathbf{y}_\ell^2))\} \\ = 4(k-1) |B_m| \left(\frac{q\lambda}{2}\right)^{-2(k-1)} \left(\frac{p\lambda}{2}\right)^{-2(\ell-1)} k! (\ell!)^2 \\ \times \int_{\mathbb{R}^{k-2}} da_1 \cdots da_{k-2} \int_{q\lambda L(\lambda)}^\infty da_{k-1} \exp\left\{-\max_{1 \leq i, j \leq k} |a_i - a_j|\right\}.$$

Since $a_k = 0$, we have the inequality $\max_{1 \leq i, j \leq k} |a_i - a_j| \geq \frac{1}{2} \max_{\substack{1 \leq i, j \leq k \\ i, j \neq k-1}} |a_i - a_j| + \frac{1}{2} |a_{k-1}|$, which we use to obtain

$$\begin{aligned}
& \int_{\mathbb{R}^{k-2}} da_1 \cdots da_{k-2} \int_{q\lambda L(\lambda)}^{\infty} da_{k-1} \exp\left\{-\max_{1 \leq i, j \leq k} |a_i - a_j|\right\} \\
& \leq 2^{k-1} \int_{\mathbb{R}^{k-1}} da_1 da_2 \cdots da_{k-2} \exp\left\{-\max_{1 \leq i, j \leq k} |a_i - a_j|\right\} \int_{\frac{1}{2}q\lambda L(\lambda)}^{\infty} da_{k-1} e^{-a_{k-1}} \\
& = 2^{k-1} (k-1)! e^{-\frac{1}{2}q\lambda L(\lambda)}.
\end{aligned}$$

Hence

$$\begin{aligned}
I_1 & \leq 2^{k+1} |B_m| \lambda^{-2(m-2)} \left(\frac{p}{2}\right)^{-2(\ell-1)} \left(\frac{q}{2}\right)^{-2(k-1)} (k!)^2 (\ell!)^2 e^{-\frac{1}{2}q\lambda L(\lambda)} \\
& = o(e^{-\frac{1}{2}q\lambda L(\lambda)}) \quad \text{as } \lambda \rightarrow \infty.
\end{aligned}$$

Similarly we obtain

$$I_2 = o(e^{-\frac{1}{2}p\lambda L(\lambda)}) \quad \text{as } \lambda \rightarrow \infty.$$

Now fix $0 < \delta < 1/2$ and take $L(\lambda) = \lambda^{-1+(\delta/2)}$. The bounds obtained above for I_1 and I_2 show that (3.17) holds.

This proves Theorem 2.1(i). The second part of Theorem 2.1 is derived easily from the first part.

4 Proof of Theorem 2.2

We now prove Theorem 2.2. Towards this end we need some estimates on the areas of the unions of various parallelograms. These are presented in the next subsection. The proof of these results are given in the appendix.

4.1 Area estimates

Throughout this section we assume $0 < \alpha < \beta < \pi$.

Lemma 4.1 (i) *If $H_\alpha, H_\beta > 2H_0$, then*

$$|B_{R_0, R_\alpha}^{0, \alpha} \cup B_{R_0, R_\beta}^{0, \beta}| = 4C_{\alpha, \beta} H_0 (H_\alpha + H_\beta - H_0).$$

(ii) *If $\min\{H_\alpha, H_\beta\} \leq 2H_0$, then*

$$|B_{R_0, R_\alpha}^{0, \alpha} \cup B_{R_0, R_\beta}^{0, \beta}| = C_{\alpha, \beta} \{4H_0 \max\{H_\alpha, H_\beta\} + \min\{H_\alpha^2, H_\beta^2\}\}.$$

Next we will estimate

$$\triangle(x) = \frac{1}{C_{\alpha,\beta}} \{|B_{R_0,R_\alpha}^{0,\alpha} \cup B_{R_0,R_\beta}^{0,\beta}(x)| - |B_{R_0,R_\alpha}^{0,\alpha} \cup B_{R_0,R_\beta}^{0,\beta}|\}, \quad x \in \mathbb{R}^2. \quad (4.1)$$

Taking

$$D_{R,R'}^{\theta,\theta'} := \begin{pmatrix} R \cos \theta & R' \cos \theta' \\ R \sin \theta & R' \sin \theta' \end{pmatrix}$$

and

$$A_{R,R'}^{\theta,\theta'} := \begin{pmatrix} R' \sin \theta' & -R' \cos \theta' \\ -R \sin \theta & R \cos \theta \end{pmatrix},$$

for $\theta, \theta' \in [0, \pi)$, $R, R' > 0$, we have $B_{R_\alpha,R_\beta}^{\alpha,\beta} = D_{R_\alpha,R_\beta}^{\alpha,\beta} [-1, 1]^2$, and

$$D_{R_\alpha,R_\beta}^{\alpha,\beta}{}^{-1} = \frac{1}{\sin(\beta - \alpha) R_\alpha R_\beta} A_{R_\alpha,R_\beta}^{\alpha,\beta}.$$

In this notation we have

$$\begin{pmatrix} h_\alpha(x) \\ h_\beta(x) \end{pmatrix} = D_{\sin \beta, \sin \alpha}^{\alpha,\beta}{}^{-1} x = \frac{1}{C_{\alpha,\beta}} \begin{pmatrix} \sin \alpha \langle x, e_{\beta - \frac{\pi}{2}} \rangle \\ \sin \beta \langle x, e_{\alpha + \frac{\pi}{2}} \rangle \end{pmatrix} \quad (4.2)$$

and

$$h_0(x) := \frac{\langle x, e_{\frac{\pi}{2}} \rangle}{\sin \alpha \sin \beta} = h_\alpha(x) + h_\beta(x), \quad x \in \mathbb{R}^2.$$

where h_α , h_β and h_0 are as defined in Section 2.1. Note that we have

$$(h_\alpha(x), h_\beta(x)) \in [-H_\alpha, H_\alpha] \times [-H_\beta, H_\beta], \text{ if and only if } x \in B_{R_\alpha,R_\beta}^{\alpha,\beta}.$$

See Figure 4.

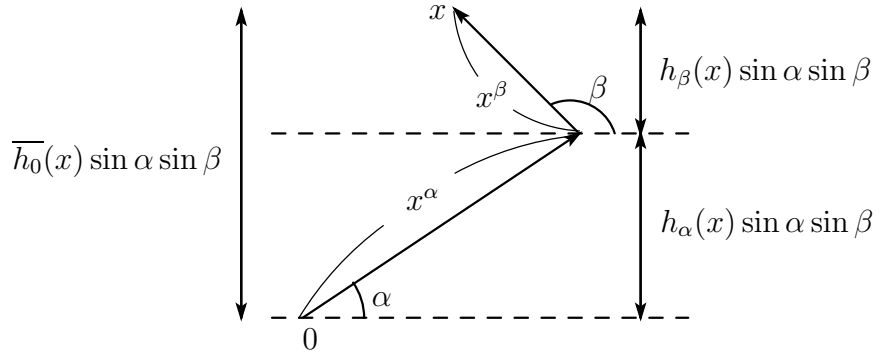


Figure 4: The quantities h_α , h_β and \overline{h}_0 .

Lemma 4.2 Assume that $x \in \mathbb{R}^2$ with $h_\alpha(x) \in [-H_\alpha, H_\alpha]$, $h_\beta(x) \in [-H_\beta, H_\beta]$.

(i) Suppose that $2H_0 < H_\alpha, H_\beta$. Then

$$\begin{aligned}\Delta(x) &= \frac{1}{2} \max\{-h_\alpha(x) + 2H_0 - H_\alpha, h_\beta(x) + 2H_0 - H_\beta, 0\}^2 \\ &\quad + \frac{1}{2} \max\{h_\alpha(x) + 2H_0 - H_\alpha, -h_\beta(x) + 2H_0 - H_\beta, 0\}^2.\end{aligned}$$

(ii) Suppose that $2H_0 \geq \min\{H_\alpha, H_\beta\}$ and $H_\alpha \geq H_\beta$.

(a) When $|\bar{h}_0(x)| \leq H_\alpha - H_\beta$,

$$\Delta(x) = \begin{cases} h_\beta(x)^2, & \text{if } |h_\beta(x)| \leq 2H_0 - H_\beta, \\ h_\beta(x)^2 - \frac{1}{2}\{|h_\beta(x)| - (2H_0 - H_\beta)\}^2, & \text{if } |h_\beta(x)| > 2H_0 - H_\beta. \end{cases}$$

(b) When $|\bar{h}_0(x)| > H_\alpha - H_\beta$ and $|h_\beta(x)| \leq 2H_0 - H_\beta$,

$$\begin{aligned}\Delta(x) &= h_\beta(x)^2 + \frac{1}{2}\{|\bar{h}_0(x)| - (H_\alpha - H_\beta)\}^2 \\ &\quad + \{2H_0 - H_\beta - \operatorname{sgn}(\bar{h}_0(x))h_\beta(x)\}\{|\bar{h}_0(x)| - (H_\alpha - H_\beta)\}.\end{aligned}$$

(c) When $|\bar{h}_0(x)| > H_\alpha - H_\beta$, $|h_\beta(x)| > 2H_0 - H_\beta$ and $\bar{h}_0(x)h_\beta(x) > 0$,

$$\begin{aligned}\Delta(x) &= h_\beta(x)^2 - \frac{1}{2}\{|h_\beta(x)| - (2H_0 - H_\beta)\}^2 \\ &\quad + \frac{1}{2}[2H_0 - H_\alpha + \operatorname{sgn}(h_\beta(x))h_\alpha(x)]_+^2,\end{aligned}$$

where $[a]_+ = \max\{a, 0\}$, $[a]_- = \max\{-a, 0\}$.

(d) When $|\bar{h}_0(x)| > H_\alpha - H_\beta$, $|h_\beta(x)| > 2H_0 - H_\beta$ and $\bar{h}_0(x)h_\beta(x) < 0$,

$$\begin{aligned}\Delta(x) &= h_\beta(x)^2 - \frac{1}{2}\{|h_\beta(x)| - (2H_0 - H_\beta)\}^2 \\ &\quad + \{|\bar{h}_0(x)| - (H_\alpha - H_\beta)\} \\ &\quad \times [2H_0 - H_\beta + |h_\beta(x)| + \frac{1}{2}\{|\bar{h}_0(x)| - (H_\alpha - H_\beta)\}].\end{aligned}$$

Remark 4.1. The area $\{x \in \mathbb{R}^2 : \Delta(x) = 0\}$ depends on angles α, β and needle lengths R_0, R_α, R_β . From the above lemma we see that

$$\{x \in \mathbb{R}^2 : \Delta(x) = 0\} = B_{R_\alpha - 2R_0^\alpha, R_\beta - 2R_0^\beta}^{\alpha, \beta}, \quad \text{when } 2H_0 < H_\alpha, H_\beta, \quad (4.3)$$

and

$$\{x \in \mathbb{R}^2 : \Delta(x) = 0\} = B_{[R_\alpha - R_\beta]_+, [R_\beta - R_\alpha]_+}^{\alpha, \beta}, \quad \text{when } 2H_0 \geq \min\{H_\alpha, H_\beta\}, \quad (4.4)$$

where for $\theta = 0, \alpha, \beta$, $R_\theta^0 = H_\theta \sin(\beta - \alpha)$, $R_\theta^\alpha = H_\theta \sin \beta$, $R_\theta^\beta = H_\theta \sin \alpha$. In particular $R_\theta^0 = R_\theta$.

Since

$$A_{R_\alpha, R_\beta}^{\alpha, \beta} x = \begin{pmatrix} R_\beta \langle x, e_{\beta - \frac{\pi}{2}} \rangle \\ R_\alpha \langle x, e_{\alpha + \frac{\pi}{2}} \rangle \end{pmatrix},$$

we have

$$\begin{aligned} M(A_{R_\alpha, R_\beta}^{\alpha, \beta} \mathbf{x}_k(0)) &= R_\beta M(\mathbf{x}_k(\beta - \frac{\pi}{2})) = C_{\alpha, \beta} H_\beta M(h_\alpha(\mathbf{x}_k)), \\ M(A_{R_\alpha, R_\beta}^{\alpha, \beta} \mathbf{x}_k(\frac{\pi}{2})) &= R_\alpha M(\mathbf{x}_k(\alpha + \frac{\pi}{2})) = C_{\alpha, \beta} H_\alpha M(h_\beta(\mathbf{x}_k)). \end{aligned}$$

For $\mathbf{x}_k \in \mathbb{R}^{2^k}$, $\mathbf{y}_\ell \in \mathbb{R}^{2^\ell}$ and $u \in \mathbb{R}^2$ we write

$$\mathbf{x}_k \cdot \mathbf{y}_\ell = (x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_\ell) \in (\mathbb{R}^2)^{k+\ell},$$

and $\mathbf{x}_k + u = (x_1 + u, x_2 + u, \dots, x_k + u) \in (\mathbb{R}^2)^k$. We put

$$\Delta(\mathbf{x}_k, \mathbf{y}_\ell | u) = \frac{1}{C_{\alpha, \beta}} \{|B_{R_0, R_\alpha}^{0, \alpha}(\mathbf{x}_k) \cup B_{R_0, R_\beta}^{0, \beta}(\mathbf{y}_\ell + u)| - |B_{R_0, R_\alpha}^{0, \alpha} \cup B_{R_0, R_\beta}^{0, \beta}(u)|\},$$

and write $\Delta(\mathbf{x}_k, \mathbf{y}_\ell)$ for $\Delta(\mathbf{x}_k, \mathbf{y}_\ell | \mathbf{0})$. The following two lemmas are important to show the main theorem. Their proofs are given in the appendix.

Lemma 4.3 *Let $\mathbf{x}_k \in (\mathbb{R}^2)^k$ with $x_k = \mathbf{0}$ and $\mathbf{y}_\ell \in (\mathbb{R}^2)^\ell$ with $y_\ell = \mathbf{0}$.*

(i) *Suppose that $2H_0 < H_\alpha, H_\beta$. If*

$$\begin{aligned} M(h_\alpha(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell)) &< H_\alpha - 2H_0 \quad \text{and} \\ M(h_\beta(\mathbf{x}_k)) + M(h_\beta(\mathbf{y}_\ell)) &< H_\beta - 2H_0 \end{aligned} \tag{4.5}$$

hold, then we have

$$\begin{aligned} \Delta(\mathbf{x}_k, \mathbf{y}_\ell) &\leq \frac{1}{C_{\alpha, \beta}} \{|B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha}(\mathbf{x}_k) \setminus B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha}| + |B_{R_0, R_\beta - R_0^\beta}^{0, \beta}(\mathbf{y}_\ell) \setminus B_{R_0, R_\beta - R_0^\beta}^{0, \beta}|\}, \\ \Delta(\mathbf{x}_k, \mathbf{y}_\ell) &\geq \frac{1}{C_{\alpha, \beta}} \{|B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha}(\mathbf{x}_k) \setminus B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha}| + |B_{R_0, R_\beta - R_0^\beta}^{0, \beta}(\mathbf{y}_\ell) \setminus B_{R_0, R_\beta - R_0^\beta}^{0, \beta}|\} \\ &\quad - M(h_\alpha(\mathbf{y}_\ell))M(h_\beta(\mathbf{x}_k)). \end{aligned}$$

(ii) *Suppose that $2H_0 \geq \min\{H_\alpha, H_\beta\}$ and $H_\alpha > H_\beta$. If $M(h_\alpha(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell)) < H_\alpha - H_\beta$ and $M(h_\beta(\mathbf{x}_k)) + M(h_\beta(\mathbf{y}_\ell)) < H_\beta$ hold, then we have*

$$\begin{aligned} \Delta(\mathbf{x}_k, \mathbf{y}_\ell) &\leq \frac{1}{C_{\alpha, \beta}} \{|B_{R_0, R_\alpha - \frac{1}{2}R_\beta^\alpha}^{0, \alpha}(\mathbf{x}_k) \setminus B_{R_0, R_\alpha - \frac{1}{2}R_\beta^\alpha}^{0, \alpha}| + |B_{\frac{1}{2}R_\beta^0, \frac{1}{2}R_\beta}^{0, \beta}(\mathbf{y}_\ell) \setminus B_{\frac{1}{2}R_\beta^0, \frac{1}{2}R_\beta}^{0, \beta}|\} \\ &\quad + \frac{1}{2}M(h_\beta(\mathbf{x}_k))^2 + \frac{1}{2}M(h_\alpha(\mathbf{y}_\ell))^2, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \Delta(\mathbf{x}_k, \mathbf{y}_\ell) &\geq \frac{1}{C_{\alpha, \beta}} \{|B_{R_0, R_\alpha - \frac{1}{2}R_\beta^\alpha}^{0, \alpha}(\mathbf{x}_k) \setminus B_{R_0, R_\alpha - \frac{1}{2}R_\beta^\alpha}^{0, \alpha}| + |B_{\frac{1}{2}R_\beta^0, \frac{1}{2}R_\beta}^{0, \beta}(\mathbf{y}_\ell) \setminus B_{\frac{1}{2}R_\beta^0, \frac{1}{2}R_\beta}^{0, \beta}|\} \\ &\quad - M(h_\beta(x_k))M(h_\beta(y_\ell)) - M(h_\beta(x_k))M(h_\alpha(y_\ell)) \\ &\quad - (M(h_\beta(x_k)))^2 - (M(h_\alpha(y_\ell)))^2. \end{aligned}$$

(iii) Suppose that $2H_0 \geq H_\alpha = H_\beta$. If $M(h_\alpha(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell)) < H_\alpha$ and $M(h_\beta(\mathbf{x}_k)) + M(h_\beta(\mathbf{y}_\ell)) < H_\beta$ hold, then we have

$$\begin{aligned} \Delta(\mathbf{x}_k, \mathbf{y}_\ell) &\leq \frac{1}{C_{\alpha,\beta}} \{ |B_{\frac{1}{2}R_\alpha^0, \frac{1}{2}R_\alpha}^{0,\alpha}(\mathbf{x}_k) \setminus B_{\frac{1}{2}R_\alpha^0, \frac{1}{2}R_\alpha}^{0,\alpha}| + |B_{\frac{1}{2}R_\beta^0, \frac{1}{2}R_\beta}^{0,\beta}(\mathbf{y}_\ell) \setminus B_{\frac{1}{2}R_\beta^0, \frac{1}{2}R_\beta}^{0,\beta}| \} \\ &\quad + (2H_0 - H_\beta)M(\bar{h}_0(\mathbf{x}_k \cdot \mathbf{y}_\ell)) + \frac{1}{2}M(h_\beta(\mathbf{x}_k))^2 + \frac{1}{2}M(h_\alpha(\mathbf{y}_\ell))^2, \end{aligned}$$

and

$$\begin{aligned} \Delta(\mathbf{x}_k, \mathbf{y}_\ell) &\geq \frac{1}{C_{\alpha,\beta}} \{ |B_{\frac{1}{2}R_\alpha^0, \frac{1}{2}R_\alpha}^{0,\alpha}(\mathbf{x}_k) \setminus B_{\frac{1}{2}R_\alpha^0, \frac{1}{2}R_\alpha}^{0,\alpha}| + |B_{\frac{1}{2}R_\beta^0, \frac{1}{2}R_\beta}^{0,\beta}(\mathbf{y}_\ell) \setminus B_{\frac{1}{2}R_\beta^0, \frac{1}{2}R_\beta}^{0,\beta}| \} \\ &\quad + (2H_0 - H_\beta)M(\bar{h}_0(\mathbf{x}_k \cdot \mathbf{y}_\ell)) - \frac{1}{2}M(\bar{h}_0(\mathbf{x}_k \cdot \mathbf{y}_\ell))^2 \\ &\quad - \min\{M(\bar{h}_0(\mathbf{x}_k)), M(\bar{h}_0(\mathbf{y}_\ell))\} \{M(h_\beta(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell))\}. \end{aligned} \quad (4.7)$$

Lemma 4.4 Let $\mathbf{x}_k \in (\mathbb{R}^2)^k$ with $x_k = \mathbf{0}$, $\mathbf{y}_\ell \in (\mathbb{R}^2)^\ell$ with $y_\ell = \mathbf{0}$ and $u \in \mathbb{R}^2$.

(i) Suppose that $2H_0 < H_\alpha, H_\beta$. If $M(h_\alpha(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell)) + |h_\alpha(u)| < H_\alpha - 2H_0$ and $M(h_\beta(\mathbf{x}_k)) + M(h_\beta(\mathbf{y}_\ell)) + |h_\beta(u)| < H_\beta - 2H_0$ hold, then we have

$$\Delta(\mathbf{x}_k, \mathbf{y}_\ell | u) = \Delta(\mathbf{x}_k, \mathbf{y}_\ell).$$

(ii) Suppose that $2H_0 \geq \min\{H_\alpha, H_\beta\}$ and $H_\alpha > H_\beta$. If $M(h_\alpha(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell)) + |h_\alpha(u)| < H_\alpha - H_\beta$ and $M(h_\beta(\mathbf{x}_k)) + M(h_\beta(\mathbf{y}_\ell)) + |h_\beta(u)| < H_\beta$ hold, then we have

$$|\Delta(\mathbf{x}_k, \mathbf{y}_\ell | u) - \Delta(\mathbf{x}_k, \mathbf{y}_\ell)| \leq h_\beta(u)^2.$$

(iii) Suppose that $2H_0 \geq H_\alpha = H_\beta$. If $M(h_\alpha(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell)) + |h_\alpha(u)| < H_\alpha$ and $M(h_\beta(\mathbf{x}_k)) + M(h_\beta(\mathbf{y}_\ell)) + |h_\beta(u)| < H_\beta$ hold, then we have

$$\begin{aligned} &| \Delta(\mathbf{x}_k, \mathbf{y}_\ell | u) - \Delta(\mathbf{x}_k, \mathbf{y}_\ell) | \\ &\quad - (2H_0 - H_\beta) \{ M(\bar{h}_0(\mathbf{x}_k \cdot (\mathbf{y}_\ell + u))) - |\bar{h}_0(u)| - M(\bar{h}_0(\mathbf{x}_k \cdot \mathbf{y}_\ell)) \} | \\ &\leq h_\alpha(u)^2 + h_\beta(u)^2 + |M(\bar{h}_0(\mathbf{x}_k \cdot (\mathbf{y}_\ell + u))) - |\bar{h}_0(u)| - M(\bar{h}_0(\mathbf{x}_k \cdot \mathbf{y}_\ell))| \\ &\quad \times \{ M(h_\alpha(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell)) + |h_\alpha(u)| + M(h_\beta(\mathbf{x}_k)) + M(h_\beta(\mathbf{y}_\ell)) + |h_\beta(u)| \}. \end{aligned}$$

4.2 The asymptotic shape

First, we examine the behaviour of the function $\mu_{\lambda\rho}(C_0 \in \Lambda(\mathbf{k})|\Gamma_0)$ as $\lambda \rightarrow \infty$ when $\mathbf{k} = (0, k_\alpha, k_\beta)$. When $\mathbf{k} = (k_0, k_\alpha, 0)$ or $\mathbf{k} = (k_0, 0, k_\beta)$, we can estimate similarly. From (3.1) we have

$$\mu_{\lambda\rho}(C_0 \in \Lambda(0, k_\alpha, k_\beta) | \Gamma_0) = \lambda^{|\mathbf{k}|-1} |\mathbf{k}| \frac{p_\alpha^{k_\alpha} p_\beta^{k_\beta}}{(k_\alpha)! k_\beta!} F_\lambda(0, k_\alpha, k_\beta), \quad (4.8)$$

where

$$\begin{aligned} F_\lambda(0, k_\alpha, k_\beta) &= \int_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{y}_{k_\alpha-1} \int_{(\mathbb{R}^2)^{k_\beta}} d\mathbf{z}_{k_\beta} 1_{\Lambda(0, k_\alpha, k_\beta)}(C_0(\mathbf{y}_{k_\alpha}, \mathbf{z}_{k_\beta})) \\ &\quad \times e^{-\lambda \{ p_0 |B_{R_\alpha, R_0}^{\alpha, 0}(\mathbf{y}_{k_\alpha}) \cup B_{R_\beta, R_0}^{\beta, 0}(\mathbf{z}_{k_\beta})| + p_\alpha |B_{R_\beta, R_\alpha}^{\beta, \alpha}(\mathbf{z}_{k_\beta})| + p_\beta |B_{R_\alpha, R_\beta}^{\alpha, \beta}(\mathbf{y}_{k_\alpha})| \}}. \end{aligned}$$

We put

$$\begin{aligned}\Phi(\mathbf{p}) &= p_0 |B_{R_\alpha, R_0}^{\alpha, 0} \cup B_{R_\beta, R_0}^{\beta, 0}| + p_\alpha |B_{R_\beta, R_\alpha}^{\beta, \alpha}| + p_\beta |B_{R_\alpha, R_\beta}^{\alpha, \beta}|, \\ f_\lambda(0, k_\alpha, k_\beta) &= F_\lambda(0, k_\alpha, k_\beta) e^{\lambda \Phi(\mathbf{p})}.\end{aligned}\tag{4.9}$$

To examine the function $f_\lambda(\mathbf{k})$, we introduce the following functions

$$\begin{aligned}\chi_c^{\theta_1, \theta_2, \theta_3}(\mathbf{x}, \mathbf{y}|z) &= e^{-c\{|B_{R_{\theta_1}, R_{\theta_2}}^{\theta_1, \theta_2}(\mathbf{x}) \cup B_{R_{\theta_1}, R_{\theta_3}}^{\theta_1, \theta_3}(\mathbf{y}+z)| - |B_{R_{\theta_1}, R_{\theta_2}}^{\theta_1, \theta_2} \cup B_{R_{\theta_1}, R_{\theta_3}}^{\theta_1, \theta_3}(z)|\}}, \\ \chi_c^{\theta_1, \theta_2}(\mathbf{x}) &= e^{-c\{|B_{R_{\theta_1}, R_{\theta_2}}^{\theta_1, \theta_2}(\mathbf{x})| - |B_{R_{\theta_1}, R_{\theta_2}}^{\theta_1, \theta_2}|\}},\end{aligned}\tag{4.10}$$

for $\theta_1, \theta_2, \theta_3 \in [0, \pi)$, $c > 0$, $\mathbf{x} \in (\mathbb{R}^2)^k$, $\mathbf{y} \in (\mathbb{R}^2)^{k'}$, $k, k' \in \mathbb{N}$ and $z \in \mathbb{R}^2$. We write $\chi_c^{\theta_1, \theta_2, \theta_3}(\mathbf{x}, \mathbf{y})$ for $\chi_c^{\theta_1, \theta_2, \theta_3}(\mathbf{x}, \mathbf{y}|\mathbf{0})$. By using these functions we obtain

$$\begin{aligned}f_\lambda(0, k_\alpha, k_\beta) &= \int_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{y}_{k_\alpha-1} \int_{(\mathbb{R}^2)^{k_\beta}} d\mathbf{z}_{k_\beta} 1_{\Lambda(0, k_\alpha, k_\beta)}(C_{\mathbf{0}}(\mathbf{y}_{k_\alpha}, \mathbf{z}_{k_\beta})) \\ &\quad \times \chi_{\lambda p_0}^{0, \alpha, \beta}(\mathbf{y}_{k_\alpha}, \mathbf{z}_{k_\beta}) \chi_{\lambda p_\alpha}^{\alpha, \beta}(\mathbf{z}_{k_\beta}) \chi_{\lambda p_\beta}^{\alpha, \beta}(\mathbf{y}_{k_\alpha}).\end{aligned}$$

Putting $\mathbf{u}_{k_\alpha} = \mathbf{y}_{k_\alpha} - \mathbf{y}_{k_\alpha}$, $\mathbf{v}_{k_\beta} = \mathbf{z}_{k_\beta} - \mathbf{z}_{k_\beta}$ and $z_{k_\beta} = z$, we have

$$f_\lambda(0, k_\alpha, k_\beta) = \int_{\mathbb{R}^2} dz g_\lambda(0, k_\alpha, k_\beta, z) \chi_{\lambda p_0}^{0, \alpha, \beta}(\mathbf{0}, z),$$

where

$$\begin{aligned}g_\lambda(0, k_\alpha, k_\beta, z) &= \int_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{u}_{k_\alpha-1} \int_{(\mathbb{R}^2)^{k_\beta-1}} d\mathbf{v}_{k_\beta-1} 1_{\Lambda(0, k_\alpha, k_\beta)}(C_{\mathbf{0}}(\mathbf{u}_{k_\alpha}, \mathbf{v}_{k_\beta} + z)) \\ &\quad \times \chi_{\lambda p_0}^{0, \alpha, \beta}(\mathbf{u}_{k_\alpha}, \mathbf{v}_{k_\beta} | z) \chi_{\lambda p_\alpha}^{\alpha, \beta}(\mathbf{v}_{k_\beta}) \chi_{\lambda p_\beta}^{\alpha, \beta}(\mathbf{u}_{k_\alpha}).\end{aligned}\tag{4.11}$$

Combining the above with (4.8) we have

$$\begin{aligned}&\mu_{\lambda \rho}(C_{\mathbf{0}} \in \Lambda(0, k_\alpha, k_\beta) \mid \Gamma_{\mathbf{0}}) \\ &= e^{-\lambda \Phi(\mathbf{p})} \lambda^{|\mathbf{k}|-1} |\mathbf{k}| \frac{p_\alpha^{k_\alpha} p_\beta^{k_\beta}}{k_\alpha! k_\beta!} \int_{\mathbb{R}^2} dz g_\lambda(0, k_\alpha, k_\beta, z) \chi_{\lambda p_0}^{0, \alpha, \beta}(\mathbf{0}, z).\end{aligned}\tag{4.12}$$

Remark 4.2. The function $\chi_{\lambda p_0}^{0, \alpha, \beta}$ determines the structure of finite clusters. From Remark 4.1 we see that $\chi_{\lambda p_0}^{0, \alpha, \beta}(0, z) = \exp[-\lambda p_0 C_{\alpha, \beta} \triangle(z)] = 1$ if and only if

$$\begin{aligned}z &\in B_{R_\alpha - 2R_0^\alpha, R_\beta - 2R_0^\beta}, & \text{when } H_\alpha, H_\beta > 2H_0, \\ z &\in B_{[R_\alpha - R_\beta]_+, [R_\beta - R_\alpha]_+}, & \text{when } \min\{H_\alpha, H_\beta\} \leq 2H_0.\end{aligned}$$

We divide into four cases and obtain estimates.

Case (1) $2H_0 < H_\alpha, H_\beta$. In this case we will show that

$$\begin{aligned}
& \mu_{\lambda\rho}(C_0 \in \Lambda(0, k_\alpha, k_\beta) | \Gamma_0) \\
& \sim \exp[-4C_{\alpha,\beta}\lambda\{p_0H_0(H_\alpha + H_\beta - H_0) + (1 - p_0)H_\alpha H_\beta\}] \\
& \times \left(\frac{1}{4C_{\alpha,\beta}\lambda}\right)^{|\mathbf{k}|-3} |\mathbf{k}| H_\alpha H_\beta (H_\alpha - 2H_0)(H_\beta - 2H_0) \\
& \times p_\alpha^{k_\alpha} k_\alpha! G^{k_\alpha}(p_0H_0 + p_\beta H_\beta, p_\beta H_\alpha, p_0(H_\alpha - H_0)) \\
& \times p_\beta^{k_\beta} k_\beta! G^{k_\beta}(p_\alpha H_\beta, p_0H_0 + p_\alpha H_\alpha, p_0(H_\beta - H_0)), \tag{4.13}
\end{aligned}$$

where for $c_1, c_2, c_3 > 0$

$$G^k(c_1, c_2, c_3) = \left(\frac{1}{k!}\right)^2 \int_{(\mathbb{R}^2)^{k-1}} d\mathbf{u}_{k-1} \gamma^k(c_1, c_2, c_3)(\mathbf{u}_k), \tag{4.14}$$

$$\gamma^k(c_1, c_2, c_3)(\mathbf{u}_k) = \exp[-\{c_1 M(\mathbf{u}_k^1) + c_2 M(\mathbf{u}_k^2) + c_3 M(\mathbf{u}_k^1 + \mathbf{u}_k^2)\}]. \tag{4.15}$$

From Remark 4.2 we see that the asymptotic shape of the cluster is given by

$$\{x \in \mathbb{R}^2 : |h_\alpha(x)| \leq H_\alpha - 2H_0, |h_\beta(x)| \leq H_\beta - 2H_0\}.$$

By Lemma 4.2 (i) and Lemma 4.4 (i) we have

$$f_\lambda(0, k_\alpha, k_\beta) \sim |B_{R_\alpha - 2R_0^\alpha, R_\beta - 2R_0^\beta}^{\alpha, \beta}| g_\lambda(0, k_\alpha, k_\beta), \quad \text{as } \lambda \rightarrow \infty. \tag{4.16}$$

By Lemma 4.3 (i) we have

$$\begin{aligned}
& g_\lambda(0, k_\alpha, k_\beta) \\
& \sim \int_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{u}_{k_\alpha-1} e^{-\lambda\{p_0|B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha}(\mathbf{u}_{k_\alpha}) \setminus B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha}| + p_\beta|B_{R_\alpha, R_\beta}^{\alpha, \beta}(\mathbf{u}_{k_\alpha}) \setminus B_{R_\alpha, R_\beta}^{\alpha, \beta}|\}} \\
& \times \int_{(\mathbb{R}^2)^{k_\beta-1}} d\mathbf{v}_{k_\beta-1} e^{-\lambda\{p_0|B_{R_0, R_\beta - R_0^\beta}^{0, \beta}(\mathbf{v}_{k_\beta}) \setminus B_{R_0, R_\beta - R_0^\beta}^{0, \beta}| + p_\alpha|B_{R_\alpha, R_\beta}^{\alpha, \beta}(\mathbf{v}_{k_\beta}) \setminus B_{R_\alpha, R_\beta}^{\alpha, \beta}|\}}
\end{aligned}$$

Using Lemma 3.1 and putting $\hat{\mathbf{u}} = A_{2\lambda \sin \beta, 2\lambda \sin \alpha}^{\alpha, \beta} \mathbf{u}$, by a simple calculation we have

$$\begin{aligned}
& \int_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{u}_{k_\alpha-1} e^{-\lambda\{p_0|B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha}(\mathbf{u}_{k_\alpha}) \setminus B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha}| + p_\beta|B_{R_\alpha, R_\beta}^{\alpha, \beta}(\mathbf{u}_{k_\alpha}) \setminus B_{R_\alpha, R_\beta}^{\alpha, \beta}|\}} \\
& \sim \int_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{u}_{k_\alpha-1} e^{-2C_{\alpha,\beta}\lambda[(p_0H_0 + p_\beta H_\beta)M(h_\alpha(\mathbf{u}_{k_\alpha})) + p_0(H_\alpha - H_0)M(\bar{h}_0(\mathbf{u}_{k_\alpha})) + p_\beta H_\beta M(h_\alpha(\mathbf{u}_{k_\alpha}))]} \\
& = \left(\frac{1}{4C_{\alpha,\beta}\lambda^2}\right)^{k_\alpha-1} G^{k_\alpha}(p_0H_0 + p_\beta H_\beta, p_\beta H_\alpha, p_0(H_\alpha - H_0)).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \int_{(\mathbb{R}^2)^{k_\beta-1}} d\mathbf{v}_{k_\beta-1} e^{-\lambda\{p_0|B_{R_0, R_\beta - R_0^\beta}^{0, \beta}(\mathbf{v}_{k_\beta}) \setminus B_{R_0, R_\beta - R_0^\beta}^{0, \beta}| + p_\alpha|B_{R_\alpha, R_\beta}^{\alpha, \beta}(\mathbf{v}_{k_\beta}) \setminus B_{R_\alpha, R_\beta}^{\alpha, \beta}|\}} \\
& \sim \left(\frac{1}{4C_{\alpha,\beta}\lambda^2}\right)^{k_\beta-1} G^{k_\beta}(p_\alpha H_\beta, p_0H_0 + p_\alpha H_\alpha, p_0(H_\beta - H_0))
\end{aligned}$$

Since by Lemma 4.1 (i)

$$\Phi(\mathbf{p}) = 4C_{\alpha,\beta}\{p_0H_0(H_\alpha + H_\beta - H_0) + (1 - p_0)H_\alpha H_\beta\}, \quad (4.17)$$

we have (4.13) from (4.12) and the above estimates.

Case (2) $2H_0 \geq H_\beta$, $H_\alpha > H_\beta$. In this case we will show that

$$\begin{aligned} & \mu_{\lambda\rho}(C_0 \in \Lambda(0, k_\alpha, k_\beta) | \Gamma_0) \\ & \sim \exp[-4C_{\alpha,\beta}\lambda\{p_0H_0H_\alpha + \frac{p_0}{4}H_\beta^2 + (1 - p_0)H_\alpha H_\beta\}] \\ & \times \left(\frac{1}{4C_{\alpha,\beta}\lambda}\right)^{|\mathbf{k}| - \frac{5}{2}} |\mathbf{k}| |H_\alpha - H_\beta| \left(\frac{\pi}{p_0}\right)^{\frac{1}{2}} \\ & \times p_\alpha^{k_\alpha} k_\alpha! G^{k_\alpha}(p_0H_0 + p_\beta H_\beta, p_\beta H_\alpha, p_0(H_\alpha - \frac{1}{2}H_\beta)) \\ & \times p_\beta^{k_\beta} k_\beta! G^{k_\beta}(p_\alpha H_\beta, \frac{1}{2}p_0H_\beta + p_\alpha H_\alpha, \frac{1}{2}p_0H_\beta). \end{aligned} \quad (4.18)$$

From Remark 4.2 we see that the asymptotic shape of the cluster is given by

$$\{x \in \mathbb{R}^2 : |h_\alpha(x)| \leq H_\alpha - H_\beta, |h_\beta(x)| = 0\}.$$

By Lemma 4.4 (ii) and a simple calculation we have

$$g_\lambda(0, k_\alpha, k_\beta, z) \sim g_\lambda(0, k_\alpha, k_\beta) \quad \text{as } \lambda \rightarrow \infty,$$

when $|h_\alpha(z)| < H_\alpha - H_\beta$, $|h_\beta(z)| = o(1)$. From Lemma 4.2 (ii) we have

$$\chi_{\lambda p_0}^{0,\alpha,\beta}(\mathbf{0}, z) = e^{-p_0 C_{\alpha,\beta} \lambda h_\beta(z)^2},$$

if $|\bar{h}_0(z)| \leq H_\alpha - H_\beta$, $|h_\beta(z)| \leq 2H_0 - H_\beta$. Then we have

$$\begin{aligned} f_\lambda(0, k_\alpha, k_\beta) & \sim g_\lambda(0, k_\alpha, k_\beta) \int_{\mathbb{R}^2} dz \chi_{\lambda p_0}^{0,\alpha,\beta}(\mathbf{0}, z) \\ & \sim 2|H_\alpha - H_\beta| \left(\frac{C_{\alpha,\beta}\pi}{p_0\lambda}\right)^{1/2} g_\lambda(0, k_\alpha, k_\beta) \quad \text{as } \lambda \rightarrow \infty. \end{aligned} \quad (4.19)$$

By Lemma 3.1 and Lemma 4.3 (ii) and similar calculations as above, we have

$$\begin{aligned} g_\lambda(0, k_\alpha, k_\beta) & \sim \left(\frac{1}{4C_{\alpha,\beta}\lambda^2}\right)^{k_\alpha-1} G^{k_\alpha}(p_0H_0 + p_\beta H_\beta, p_\beta H_\alpha, p_0(H_\alpha - \frac{1}{2}H_\beta)) \\ & \times \left(\frac{1}{4C_{\alpha,\beta}\lambda^2}\right)^{k_\beta-1} G^{k_\beta}(p_\alpha H_\beta, \frac{1}{2}p_0H_\beta + p_\alpha H_\alpha, \frac{1}{2}p_0H_\beta). \end{aligned}$$

Since by Lemma 3.1 (ii)

$$\Phi(\mathbf{p}) = 4C_{\alpha,\beta}\{p_0H_0H_\alpha + \frac{p_0}{4}H_\beta^2 + (1 - p_0)H_\alpha H_\beta\}, \quad (4.20)$$

we have (4.18) from (4.12) and the above estimates

Case (3) $2H_0 = H_\alpha = H_\beta$. In this case we will show that

$$\begin{aligned}
& \mu_{\lambda\rho}(C_{\mathbf{0}} \in \Lambda(0, k_\alpha, k_\beta) | \Gamma_0) \\
& \sim \exp[-4C_{\alpha,\beta}\lambda(4-p_0)H_0^2] \\
& \times \left(\frac{1}{4C_{\alpha,\beta}\lambda}\right)^{|\mathbf{k}|-2} |\mathbf{k}|^{\frac{3\pi+4}{2p_0}} \\
& \times p_\alpha^{k_\alpha} k_\alpha! G^{k_\alpha}((p_0+2p_\beta)H_0, 2p_\beta H_0, p_0 H_0) \\
& \times p_\beta^{k_\beta} k_\beta! G^{k_\beta}(2p_\alpha H_0, (p_0+2p_\alpha)H_0, p_0 H_0).
\end{aligned} \tag{4.21}$$

From Remark 4.2 we see that the asymptotic shape of the cluster is given by

$$\{x \in \mathbb{R}^2 : |h_\alpha(x)| \leq 0, |h_\beta(x)| \leq 0\} = \{\mathbf{0}\}.$$

By Lemma 4.4 (iii) and a simple calculation we have

$$g_\lambda(0, k_\alpha, k_\beta, z) \sim g_\lambda(0, k_\alpha, k_\beta) \quad \text{as } \lambda \rightarrow \infty,$$

when $|h_\alpha(z)| = o(1)$, $|h_\beta(z)| = o(1)$. From Lemma 4.2 (ii) we have

$$\chi_{\lambda p_0}^{0,\alpha,\beta}(\mathbf{0}, z) = \begin{cases} \exp[-\frac{1}{2}C_{\alpha,\beta}p_0\lambda(h_\alpha(z)^2 + h_\beta(z)^2)], & \bar{h}_0(z)h_\beta(z) > 0, \\ \exp[-\frac{1}{2}C_{\alpha,\beta}p_0\lambda h_\alpha(z)^2], & \bar{h}_0(z)h_\beta(z) > 0, \end{cases} \tag{4.22}$$

if $|h_\alpha(z)| \leq H_\alpha$, $|h_\beta(z)| \leq H_\beta$. Then we have

$$\begin{aligned}
f_\lambda(0, k_\alpha, k_\beta) & \sim g_\lambda(0, k_\alpha, k_\beta) \int_{\mathbb{R}^2} dz \chi_{\lambda p_\beta}^{0,\alpha,\beta}(\mathbf{0}, z) \\
& \sim \left(\frac{3\pi+4}{2p_0\lambda}\right) g_\lambda(0, k_\alpha, k_\beta) \quad \text{as } \lambda \rightarrow \infty.
\end{aligned} \tag{4.23}$$

By Lemma 3.1 and Lemma 4.3 (iii) and similar calculations as above, we have

$$\begin{aligned}
g_\lambda(0, k_\alpha, k_\beta) & \sim \left(\frac{1}{4C_{\alpha,\beta}\lambda^2}\right)^{k_\alpha-1} G^{k_\alpha}\left(\frac{1}{2}p_0 H_\alpha + p_\beta H_\beta, p_\beta H_\alpha, \frac{1}{2}p_0 H_\alpha\right) \\
& \times \left(\frac{1}{4C_{\alpha,\beta}\lambda^2}\right)^{k_\beta-1} G^{k_\beta}\left(p_\alpha H_\beta, \frac{1}{2}p_0 H_\beta + p_\alpha H_\alpha, \frac{1}{2}p_0 H_\beta\right).
\end{aligned}$$

Since by Lemma 3.1 (ii), $\Phi(\mathbf{p}) = 4C_{\alpha,\beta}(4-p_0)H_0^2$, we have (4.21) from (4.12) and the above estimates

Case (4) $2H_0 > H_\alpha = H_\beta$. In this case we will show that

$$\begin{aligned}
& \mu_{\lambda\rho}(C_{\mathbf{0}} \in \Lambda(0, k_\alpha, k_\beta) | \Gamma_0) \\
& \sim \exp[-4C_{\alpha,\beta}\lambda\{p_0 H_0 H_\alpha + (1 - \frac{3}{4}p_0)H_\alpha^2\}] \\
& \times \left(\frac{1}{4C_{\alpha,\beta}\lambda}\right)^{|\mathbf{k}|-\frac{3}{2}} |\mathbf{k}| \left(\frac{2\pi}{p_0}\right)^{\frac{1}{2}} p_\alpha^{k_\alpha} k_\alpha! p_\beta^{k_\beta} k_\beta! \\
& \times G_{\frac{1}{2}(2H_0-H_\alpha)}^{k_\alpha, k_\beta}\left(\left(\frac{p_0}{2} + p_\beta\right)H_\alpha, p_\beta H_\alpha, \frac{p_0}{2}H_\alpha, p_\alpha H_\alpha, \left(\frac{p_0}{2} + p_\alpha\right)H_\alpha, \frac{p_0}{2}H_\alpha\right).
\end{aligned} \tag{4.24}$$

where

$$G_z^{k,\ell}(c_1, c_2, c_3, c_4, c_5, c_6) = \left(\frac{1}{k!}\right)^2 \left(\frac{1}{\ell!}\right)^2 \int_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{u}_{k_\alpha-1} \int_{(\mathbb{R}^2)^{k_\beta-1}} d\mathbf{v}_{k_\beta-1} \\ \times J_z(\mathbf{u}_{k_\alpha}, \mathbf{v}_{k_\beta}) \gamma(c_1, c_2, c_3)(\mathbf{u}_{k_\alpha}) \gamma(c_4, c_5, c_6)(\mathbf{v}_{k_\beta}),$$

From Remark 4.2 we see that the asymptotic shape of the cluster is given by

$$\{x \in \mathbb{R}^2 : |h_\alpha(x)| \leq 0, |h_\beta(x)| \leq 0\} = \{\mathbf{0}\}.$$

By Lemma 4.3 (iii), Lemma 4.4 (iii) and a simple calculation we have

$$\triangle(\mathbf{x}_k, \mathbf{y}_\ell | z) \sim \frac{1}{C_{\alpha,\beta}} \{ |B_{\frac{1}{2}R_\alpha^0, \frac{1}{2}R_\alpha}^{0,\alpha}(\mathbf{x}_k) \setminus B_{\frac{1}{2}R_\alpha^0, \frac{1}{2}R_\alpha}^{0,\alpha}| + |B_{\frac{1}{2}R_\beta^0, \frac{1}{2}R_\beta}^{0,\beta}(\mathbf{y}_\ell) \setminus B_{\frac{1}{2}R_\beta^0, \frac{1}{2}R_\beta}^{0,\beta}| \} \\ + (2H_0 - H_\beta) \{ M(\bar{h}_0(\mathbf{x}_k \times (\mathbf{y}_\ell + z))) - \bar{h}_0(z) \}$$

when $|h_\alpha(z)| = o(1)$, $|h_\beta(z)| = o(1)$. From Lemma 4.2 (ii)

$$\triangle(z) = \frac{1}{2}(h_\alpha(z)^2 + h_\beta(z)^2) + (2H_0 - H_\beta)|\bar{h}_0(z)|,$$

if $|h_\alpha(z)| \leq H_\alpha$, $|h_\beta(z)| \leq H_\beta$. Then

$$f_\lambda(0, k_\alpha, k_\beta) \sim \int_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{u}_{k_\alpha-1} \int_{(\mathbb{R}^2)^{k_\beta-1}} d\mathbf{v}_{k_\beta-1} K_\lambda(\mathbf{u}_{k_\alpha}, \mathbf{v}_{k_\beta}) \\ \times \chi_{\lambda p_0}^{0,\alpha,\beta}(\mathbf{u}_{k_\alpha}, \mathbf{v}_{k_\beta}) \chi_{\lambda p_\alpha}^{\alpha,\beta}(\mathbf{v}_{k_\beta}) \chi_{\lambda p_\beta}^{\alpha,\beta}(\mathbf{u}_{k_\alpha}),$$

where

$$K_\lambda(\mathbf{u}_{k_\alpha}, \mathbf{v}_{k_\beta}) = \int_{\mathbb{R}^2} dz \exp[-\frac{1}{2}C_{\alpha,\beta}p_0\lambda(h_\alpha(z)^2 + h_\beta(z)^2)] \\ \times \exp[-\lambda C_{\alpha,\beta}p_0(2H_0 - H_\beta)M(\bar{h}_0(\mathbf{u}_{k_\alpha} \cdot (\mathbf{v}_{k_\beta} + z)))].$$

By Lemma 3.1 and Lemma 4.3 (iii) and similar calculations as above, we have

$$f_\lambda(0, k_\alpha, k_\beta) \sim \left(\frac{1}{4C_{\alpha,\beta}\lambda^2}\right)^{|k|-1} \left(\frac{8\pi C_{\alpha,\beta}\lambda}{p_0}\right)^{\frac{1}{2}} \\ \times \int_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{u}_{k_\alpha-1} \int_{(\mathbb{R}^2)^{k_\beta-1}} d\mathbf{v}_{k_\beta-1} J_{\frac{p_0}{2}(2H_0-H_\beta)}(\mathbf{u}_{k_\alpha}, \mathbf{v}_{k_\beta}) \\ \times \gamma^{k_\alpha}((\frac{1}{2}p_0 + p_\beta)H_\alpha, p_\beta H_\alpha, \frac{1}{2}p_0 H_\alpha) \gamma^{k_\beta}(p_\alpha H_\alpha, (\frac{1}{2}p_0 + p_\alpha)H_\alpha, \frac{1}{2}p_0 H_\alpha).$$

Since by Lemma 4.1 (ii), $\Phi(\mathbf{p}) = 4C_{\alpha,\beta}\{p_0 H_0 H_\alpha(1 - \frac{3}{4}p_0)H_\alpha^2\}$, we have (4.24) from (4.12) and the above estimates.

Proof of Theorem 2.2 First we examine the behaviour of the function $\mu_{\lambda\rho}(C_{\mathbf{0}} \in \Lambda(\mathbf{k})|\Gamma_0)$ as $\lambda \rightarrow \infty$ when $\mathbf{k} = (k_0, k_\alpha, k_\beta)$, with $k_0, k_\alpha, k_\beta \in \mathbb{N}$. From (1.3) and an argument similar to

that needed to obtain (4.1) we have

$$\mu_{\lambda\rho}(C_0 \in \Lambda(\mathbf{k}) \mid \Gamma_0) = \lambda^{|\mathbf{k}|-1} |\mathbf{k}| \frac{p_0^{k_0} p_\alpha^{k_\alpha} p_\beta^{k_\beta}}{k_0! k_\alpha! k_\beta!} F_\lambda(\mathbf{k}), \quad (4.25)$$

where

$$\begin{aligned} F_\lambda(\mathbf{k}) &= e^{-\lambda\{p_0|B_{R_\alpha, R_0}^{\alpha, 0} \cup B_{R_\beta, R_0}^{\beta, 0}| + p_\alpha|B_{R_0, R_\alpha}^{0, \alpha} \cup B_{R_\beta, R_\alpha}^{\beta, \alpha}| + p_\beta|B_{R_0, R_\beta}^{0, \beta} \cup B_{R_\alpha, R_\beta}^{\alpha, \beta}|\}} \\ &\quad \times \int_{(\mathbb{R}^2)^{k_0-1}} d\mathbf{x}_{k_0-1} \int_{(\mathbb{R}^2)^{k_\alpha}} d\mathbf{y}_{k_\alpha} \int_{(\mathbb{R}^2)^{k_\beta}} d\mathbf{z}_{k_\beta} 1_{\Lambda(\mathbf{k})}(C_0(\mathbf{x}_{k_0}, \mathbf{y}_{k_\alpha}, \mathbf{z}_{k_\beta})) \\ &\quad \times \chi_{\lambda p_0}^{0, \alpha, \beta}(\mathbf{y}_{k_\alpha}, \mathbf{z}_{k_\beta}) \chi_{\lambda p_\alpha}^{\alpha, \beta, 0}(\mathbf{z}_{k_\beta}, \mathbf{x}_{k_0}) \chi_{\lambda p_\beta}^{\beta, 0, \alpha}(\mathbf{x}_{k_0}, \mathbf{y}_{k_\alpha}). \end{aligned}$$

From the above we see that the probability that the cluster contains needles of three distinct orientations is much smaller than that of only two distinct orientations.

For **case (1)**, when $a, b \geq 2$, from (4.13), (4.21) and (4.18) we have

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \frac{-1}{4C_{\alpha, \beta} \lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(0, k, \ell) | \Gamma_0) &= p_0(a + b - 1) + (1 - p_0)ab, \\ \lim_{\lambda \rightarrow \infty} \frac{-1}{4C_{\alpha, \beta} \lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k, 0, \ell) | \Gamma_0) &= p_\alpha ab + \frac{p_\alpha}{4} + (1 - p_\alpha)b, \\ \lim_{\lambda \rightarrow \infty} \frac{-1}{4C_{\alpha, \beta} \lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k, \ell, 0) | \Gamma_0) &= p_\beta ab + \frac{p_\beta}{4} + (1 - p_\beta)a. \end{aligned}$$

Since

$$p_0(a + b - 1) + (1 - p_0)ab > \min\{p_\alpha ab + \frac{p_\alpha}{4} + (1 - p_\alpha)b, p_\beta ab + \frac{p_\beta}{4} + (1 - p_\beta)a\},$$

we obtain Theorem 2.2 (1) (i) and (ii). From (4.18) we see that

$$\mu_{\lambda\rho}(C_0 \in \Lambda(k, 0, \ell) | \Gamma_0) \exp\{\lambda\Phi(\mathbf{p})\} \sim c\lambda^{k+\ell-5/2},$$

and

$$\mu_{\lambda\rho}(C_0 \in \Lambda(k, \ell, 0) | \Gamma_0) \exp\{\lambda\Phi(\mathbf{p})\} \sim c'\lambda^{k+\ell-5/2},$$

with positive constants c and c' independent of λ . Thus we have (iii).

For **case (2)**, when $1/2 < \min\{a, b\} < 2$, $a \neq b$, $a, b \neq 1$, from (4.18) we have

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \frac{-1}{4C_{\alpha, \beta} \lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(0, k, \ell) | \Gamma_0) &= f(0, \alpha, \beta), \\ \lim_{\lambda \rightarrow \infty} \frac{-1}{4C_{\alpha, \beta} \lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k, 0, \ell) | \Gamma_0) &= f(\beta, 0, \alpha) \\ \lim_{\lambda \rightarrow \infty} \frac{-1}{4C_{\alpha, \beta} \lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k, \ell, 0) | \Gamma_0) &= f(\alpha, \beta, 0). \end{aligned}$$

Thus we obtain Theorem 2.2 (2).

For **case (3)**, when $0 < a = b < 1$, from (4.18) and (4.21) we have

$$\begin{aligned}\lim_{\lambda \rightarrow \infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(0, k, \ell) | \Gamma_0) &= p_0 a + (1 - \frac{3}{4}p_0)a^2, \\ \lim_{\lambda \rightarrow \infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k, 0, \ell) | \Gamma_0) &= \frac{1}{4}p_\alpha a^2 + a, \\ \lim_{\lambda \rightarrow \infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k, \ell, 0) | \Gamma_0) &= \frac{1}{4}p_\beta a^2 + a.\end{aligned}$$

If $p_\alpha \geq p_\beta$, $A(\alpha, \beta)$ occurs whenever

$$p_0 a + (1 - \frac{3}{4}p_0)a^2 < \frac{1}{4}p_\beta a^2 + a,$$

i.e., $a < \mathbf{l}_1(p_0, p_\alpha, p_\beta)$. Since $\mathbf{l}_1(p_0, p_\alpha, p_\beta) \geq 1$ for $p_0 \leq p_\beta$, we obtain Theorem 2.2 (3).

Finally for **case (4)**, when $1 < a = b < 2$, from (4.18) and (4.21) we have

$$\begin{aligned}\lim_{\lambda \rightarrow \infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(0, k, \ell) | \Gamma_0) &= p_0 a + (1 - \frac{3}{4}p_0)a^2, \\ \lim_{\lambda \rightarrow \infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k, 0, \ell) | \Gamma_0) &= p_\alpha a^2 + \frac{1}{4}p_\alpha + (1 - p_\alpha)a, \\ \lim_{\lambda \rightarrow \infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k, \ell, 0) | \Gamma_0) &= p_\beta a^2 + \frac{1}{4}p_\beta + (1 - p_\beta)a.\end{aligned}$$

If $p_\alpha \geq p_\beta$, we see that $A(\alpha, \beta)$ occurs whenever

$$p_0 a + (1 - \frac{3}{4}p_0)a^2 < p_\beta a^2 + \frac{1}{4}p_\beta + (1 - p_\beta)a,$$

i.e., $a < \mathbf{l}_2(p_0, p_\alpha, p_\beta)$. Since $\mathbf{l}_2(p_0, p_\alpha, p_\beta) \leq 1$ for $p_0 \geq p_\beta$, we obtain Theorem 2.2 (4).

Also for **case (4)** $a = b = 1$, from (4.18) and (4.21) we have Theorem 2.2 (5), easily.

5 Appendix

Proof of Lemma 3.1: The smallest rectangle containing the region $B_{R_0, R_{\pi/2}}^{0, \pi/2}(\mathbf{x}_k)$ has dimensions $(2R_0 + M(\mathbf{x}_k^1)) \times (2R_\alpha + M(\mathbf{x}_k^2))$ thereby yielding (3.5).

Let x_l and x_r be the leftmost and the rightmost points among x_1, \dots, x_k so that $M(\mathbf{x}_k^1) = |x_l(1) - x_r(1)|$. Now there are two rectangles, each of size $R_0 \times 2R_{\pi/2}$, one lying to the left of x_l and the other lying to the right of x_r which are part of $B_{R_0, R_{\pi/2}}^{0, \pi/2}(\mathbf{x}_k)$ and an area of $2R_{\pi/2}|x_l(1) - x_r(1)|$, composed of possibly many disjoint rectangles lying in between these two rectangles. Connectivity of $B_{R_0, R_{\pi/2}}^{0, \pi/2}(\mathbf{x}_k)$ ensures that the rectangles forming the area $2R_{\pi/2}|x_l(1) - x_r(1)|$ can be ordered such that neighbouring rectangles in this ordering share parts of their edges. Thus $|B_{R_0, R_{\pi/2}}^{0, \pi/2}(\mathbf{x}_k) \setminus B_{R_0, R_{\pi/2}}^{0, \pi/2}| \geq 2R_{\pi/2}M(\mathbf{x}_k^1)$. Similarly, considering the topmost and the bottommost points among x_1, \dots, x_k we obtain $|B_{R_0, R_{\pi/2}}^{0, \pi/2}(\mathbf{x}_k) \setminus B_{R_0, R_{\pi/2}}^{0, \pi/2}| \geq 2R_0M(\mathbf{x}_k^2)$. These two inequalities yield (3.6). The inequality (3.7) follows from the observation that these two regions of areas $2R_{\pi/2}M(\mathbf{x}_k^1)$ and $2R_0M(\mathbf{x}_k^2)$ have a region of area $M(\mathbf{x}_k^1)M(\mathbf{x}_k^2)$ in common.

The second part of the lemma for general bases follows from similar argument and is omitted. ■

Proof of Lemma 4.1 If $2H_0 \geq H_\beta$ and $H_\alpha \geq H_\beta$. Then

$$\begin{aligned} |B_{R_0, R_\alpha}^{0, \alpha} \cup B_{R_0, R_\beta}^{0, \beta}| &= |B_{R_0, R_\alpha}^{0, \alpha} \setminus B_{R_0, R_\beta}^{0, \beta}| + |B_{R_0, R_\beta}^{0, \beta} \setminus B_{R_0, R_\alpha}^{0, \alpha}| + |B_{R_0, R_\alpha}^{0, \alpha} \cap B_{R_0, R_\beta}^{0, \beta}| \\ &= 2R_0 \cdot 2R_\alpha \sin \alpha + R_\beta \sin(\pi - \beta) R_\beta \sin(\beta - \alpha) (\sin \alpha)^{-1} \\ &= C_{\alpha, \beta} (4H_0 H_\alpha + H_\beta^2). \end{aligned}$$

If $2H_0 \geq H_\alpha$ and $H_\beta \geq H_\alpha$. Then, similarly, we have

$$\begin{aligned} |B_{R_0, R_\alpha}^{0, \alpha} \cup B_{R_0, R_\beta}^{0, \beta}| &= 2R_0 \cdot 2R_\beta \sin \beta + R_\alpha \sin(\pi - \alpha) R_\alpha \sin(\beta - \alpha) (\sin \beta)^{-1} \\ &= C_{\alpha, \beta} (4H_0 H_\beta + H_\alpha^2). \end{aligned}$$

Finally if $H_\alpha, H_\beta > 2H_0$, then

$$\begin{aligned} |B_{R_0, R_\alpha}^{0, \alpha} \cup B_{R_0, R_\beta}^{0, \beta}| &= |B_{R_0, R_\alpha}^{0, \alpha}| + |B_{R_0, R_\beta}^{0, \beta}| - |B_{R_0, R_\alpha}^{0, \alpha} \cap B_{R_0, R_\beta}^{0, \beta}| \\ &= 4R_0 R_\alpha \sin \alpha + 4R_0 R_\beta \sin \beta - 4R_0^2 \sin \alpha \sin \beta (\sin(\beta - \alpha))^{-1} \\ &= 4C_{\alpha, \beta} H_0 (H_\alpha + H_\beta - H_0). \end{aligned}$$

This proves the lemma. ■

Proof of Lemma 4.2 Suppose that $2H_0 \geq H_\beta$ and $H_\alpha \geq H_\beta$. Also assume that $|\bar{h}_0(x)| \leq H_\alpha - H_\beta$ and $|h_\beta(x)| \leq 2H_0 - H_\beta$. In this case we have $B_{R_0, R_\alpha}^{0, \alpha} \cup B_{R_0, R_\beta}^{0, \beta}$ represented as the union of the two parallelograms $ABCD$ and $EFGH$ in Figure 5, while $B_{R_0, R_\alpha}^{0, \alpha} \cup B_{R_0, R_\beta}^{0, \beta}(x)$ is the union of $ABCD$ and $IJKL$. The difference between these two regions is thus the difference

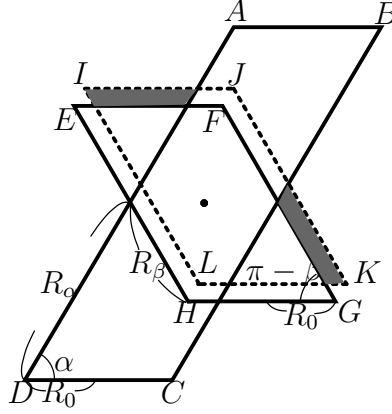


Figure 5: *Figure accompanying proof of Lemma 4.2.*

of the “dashed” triangles and the “solid” triangles outside the parallelogram $ABCD$. It is easily seen that the sum of the area of the “dashed” triangles is $\frac{\sin \alpha \sin \beta}{\sin(\beta - \alpha)} \left[\frac{R_\beta^2 \sin^2(\beta - \alpha)}{\sin^2 \alpha} + (x_1 - \frac{x_2}{\tan \alpha}) \right]$, while the sum of the areas of the solid triangles is $\frac{R_\beta^2 \sin \beta \sin(\beta - \alpha)}{\sin \alpha}$. This proves the first case Lemma 4.2 (i). By considering similar figures, the other parts of the lemma follow. \blacksquare

Proof of Lemma 4.3 First we consider the situation when $y_1 = \mathbf{0}$, $\ell = 1$ and $k = 2$ with $x_2 = \mathbf{0}$ and x_1 such that

$$|x_1^\alpha| \leq R_\alpha - 2R_0^\alpha, \quad |x_1^\beta| \leq R_\beta - 2R_0^\beta. \quad (5.26)$$

We note here that this choice of x_1 ensures the existence of the hatched region in Figure 6 which is isomorphic to a parallelogram with sides making angles α and β with the x -axis.

From Figure 6 we see that if we collapse the lines AD and BC into one and remove the parallelogram contained between these lines then each of the parallelograms $B_{R_0, R_\alpha}^{0, \alpha}$ and $B_{R_0, R_\alpha}^{0, \alpha}(x_1)$ become isomorphic to $B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha}$. Moreover $(B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha}(x_1, x_2) \setminus B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha})$ is isomorphic to $((B_{R_0, R_\alpha}^{0, \alpha}(x_1, x_2) \cup B_{R_0, R_\beta}^{0, \beta}(y_1)) \setminus (B_{R_0, R_\alpha}^{0, \alpha} \cup B_{R_0, R_\beta}^{0, \beta}))$, the shaded area.

Since $(B_{R_0, R_\alpha}^{0, \alpha} \cup B_{R_0, R_\beta}^{0, \beta}) \subseteq (B_{R_0, R_\alpha}^{0, \alpha}(x_1, x_2) \cup B_{R_0, R_\beta}^{0, \beta})$ and $B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha}(x_1, x_2) \supseteq B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha}$ we have

$$C_{\alpha, \beta} \Delta(\mathbf{x}_2, y_1) = |B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha}(\mathbf{x}_2) \setminus B_{R_0, R_\alpha - R_0^\alpha}^{0, \alpha}|. \quad (5.27)$$

Now observe that a similar result may be obtained when $x_1 = \mathbf{0}$, $k = 1$ and $\ell = 2$, $y_2 = \mathbf{0}$ and y_1 such that

$$|y_1^\alpha| \leq R_\alpha - 2R_0^\alpha, \quad |y_1^\beta| \leq R_\beta - 2R_0^\beta. \quad (5.28)$$

In this case we obtain

$$C_{\alpha, \beta} \Delta(x_1, \mathbf{y}_2) = |B_{R_0, R_\beta - R_0^\beta}^{0, \beta}(\mathbf{y}_2) \setminus B_{R_0, R_\beta - R_0^\beta}^{0, \beta}|. \quad (5.29)$$

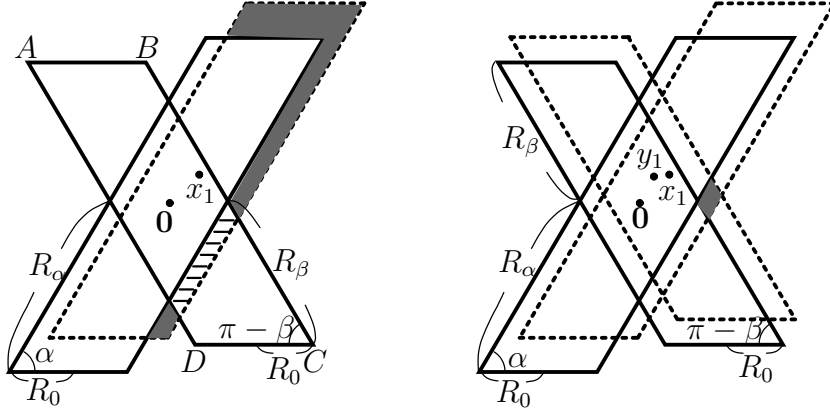


Figure 6: The two shaded regions in the figure on the left combine on collapsing the lines AD and BC . The shaded parallelogram in the figure on the right is double counted.

In case both $k = 2$ and $\ell = 2$ with x_1 and y_1 satisfying (5.26) and (5.28) we see from Figure 6 that if we add the areas obtained in (5.27) and (5.29) there is double counting of the shaded parallelogram with sides of length $|x_1^\beta|$ and $|y_1^\alpha|$ and area $|x_1^\alpha||y_1^\beta|\sin(\beta - \alpha)$. Thus we have $C_{\alpha,\beta}\Delta(\mathbf{x}_2, \mathbf{y}_2) = |B_{R_0, R_\alpha - R_\alpha^0}^{0,\alpha}(\mathbf{x}_2) \setminus B_{R_0, R_\alpha - R_\alpha^0}^{0,\alpha}| + |B_{R_0, R_\beta - R_\beta^0}^{0,\beta}(\mathbf{y}_2) \setminus B_{R_0, R_\beta - R_\beta^0}^{0,\beta}| - |x_1^\beta||y_1^\alpha|\sin(\beta - \alpha)$.

In general, for any k and ℓ , we see that if

$$M(\mathbf{x}_k) \leq R_\alpha - 2R_\alpha^0, \text{ and } M(\mathbf{y}_\ell) \leq R_\beta - 2R_\beta^0 \quad (5.30)$$

there will be many such shaded areas which will be double counted. These areas need not be all distinct and the total area of this double counted region is at most $M(\mathbf{x}_k^\beta)M(\mathbf{y}_\ell^\alpha)\sin(\beta - \alpha)$. Now note that the condition (4.5) guarantees that (5.30) holds. Hence Lemma 4.3 (i) follows.

The remaining parts of the lemmas follow from similar arguments and are explained through Figures 7 and 8. ■

Lemma 4.4 follows similarly and its proof is omitted.

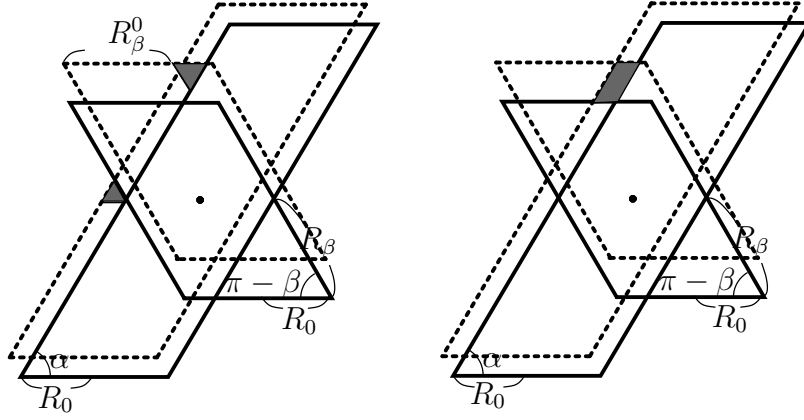


Figure 7: The shaded triangles in the left figure give the last two terms in (4.6), while the shaded parallelogram in the right figure is double counted.

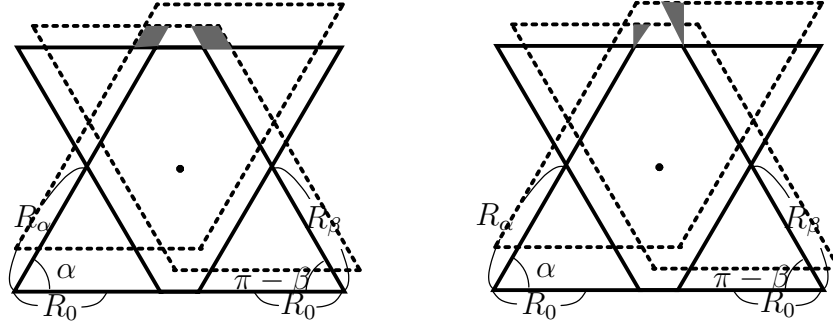


Figure 8: The shaded areas are double counted and is deducted in (4.7).

6 Acknowledgement

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